PROCEEDINGS

OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

VOL. 67 NOVEMBER, 1941 NO. 9

TECHNICAL PAPERS

AND

DISCUSSIONS

Published monthly, except July and August, at Prince and Lemon Streets, Lancaster, Pa., by the American Society of Civil Engineers. Editorial and General Offices at 33 West Thirty-ninth Street, New York, N. Y. Reprints from this publication may be made on condition that the full title of Paper, name of Author, page reference, and date of publication by the Society, are given.

Entered as Second-Class Matter, September 23, 1937, at the Post Office at Lancaster, Pa., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized on July 5, 1918.

Subscription (if entered before January 1) $8.00 per annum. Price $1.00 per copy

Copyright, 1941, *by the* AMERICAN SOCIETY OF CIVIL ENGINEERS

Printed in the United States of America

CURRENT PAPERS AND DISCUSSIONS

		Discussion closes
Reliability of Station-Year Rainfall-Frequency Determinations. *Katherine Clarke-Hafstad*	Nov., 1940	
Discussion	Jan., Feb., Mar., Apr., May, June, 1941	Closed*
Fort Peck Slide. *T. A. Middlebrooks*	Dec., 1940	
Discussion	Mar., Apr., May, Sept., 1941	Closed*
Analysis of Statically Indeterminate Trussed Structures. *O. T. Voodhigula*	Jan., 1941	
Discussion	Mar., Apr., May, Nov., 1941	Closed
Cavitation in Outlet Conduits of High Dams. *Harold A. Thomas and Emil P. Schuleen*	Nov., 1940	
Discussion	Mar., Apr., May, June, Oct., Nov., 1941	Closed
Laboratory Investigations of Soils at Flushing Meadow Park. *Donald M. Burmister*	Jan., 1941	
Discussion	May, June, Sept., Oct., Nov., 1941	Closed
Rigid Frames Without Diagonals (The Vierendeel Truss). *Louis Baes*	Jan., 1941	
Discussion	June, Sept., 1941	Closed*
Hydraulics of Sprinkling Systems for Irrigation. *J. C. Christiansen*	Jan., 1941	
Discussion	Apr., June, Nov., 1941	Closed
Concrete in Sea Water: A Revised Viewpoint Needed. *Homer M. Hadley*	Jan., 1941	
Discussion	Mar., Apr., May, June, Oct., Nov., 1941	Closed*
Value of Public Works. *J. P. Hallihan*	Feb., 1941	
Discussion	Apr., May, June, Sept., 1941	Closed*
On the Method of Complementary Energy. *H. M. Westergaard*	Feb., 1941	
Discussion	Apr., Sept., 1941	Closed*
Crack Prevention Program, Hiwassee Dam. *O. Laurgaard*	Mar., 1941	
Discussion	Sept., 1941	Closed*
Formulas for the Transportation of Bed Load. *H. A. Einstein*	Mar., 1941	
Discussion	June, Sept., 1941	Closed*
Design of Acceleration and Deceleration Lanes. *Adolphus Mitchell*	Mar., 1941	
Discussion	May, June, Sept., 1941	Closed*
Simplified Theory of the Self-Anchored Suspension Bridge. *C. H. Gronquist*	Feb., 1941	
Discussion	May, Sept., 1941	Dec., 1941
Some Economics of Airports. *W. Watters Pagon*	Feb., 1941	
Discussion	Sept., 1941	Dec., 1941
Missouri River Slope and Sediment. *William Whipple, Jr.*	Mar., 1941	
Discussion	Sept., Nov., 1941	Dec., 1941
Analysis of Building Frames with Semi-Rigid Connections. *Bruce Johnston and Edward H. Mount*	Mar., 1941	
Discussion	May, June, Sept., Oct., 1941	Dec., 1941
The Suspension Bridge Tower Cantilever Problem. *Blair Birdsall*	Apr., 1941	
Discussion	Sept., 1941	Dec., 1941
Surface Runoff Determination from Rainfall Without Using Coefficients. *W. W. Horner and S. W. Jens*	Apr., 1941	
Discussion	June, Oct., 1941	Dec., 1941
Operation Experiences, Tygart Reservoir. *Robert M. Morris and Thomas L. Reilly*	Apr., 1941	
Discussion	Oct., 1941	Dec., 1941
Consumptive Use of Water for Agriculture. *Robert L. Lowry, Jr., and Arthur F. Johnson*	Apr., 1941	
Discussion	June, Sept., Oct., 1941	Dec., 1941
Tunnel Construction, Sixth Avenue Subway, New York, N. Y. *Jacob Feld*	Apr., 1941	
Discussion	Nov., 1941	Dec., 1941
Moments in Continuous Rectangular Slabs on Rigid Supports. *L. C. Maugh and C. W. Pan*	May, 1941	
Discussion	June, Sept., 1941	Dec., 1941
The Surveyor and the Law. *A. H. Holt*	May, 1941	
Discussion	Oct., 1941	Dec., 1941
Compaction of Cohesionless Foundation Soils by Explosives. *A. K. B. Lyman*	May, 1941	
Discussion	Oct., 1941	Dec., 1941
Extensometer Stress Measurements, North Avenue Bridge, Chicago, Ill. *Lawrence T. Smith and Paul Lillard*	May, 1941	
Discussion	June, Oct., 1941	Dec., 1941
Evaluation of Flood Losses and Benefits. *Edgar E. Foster*	May, 1941	
Discussion	Sept., Oct., 1941	Dec., 1941
Traffic Engineering as Applied to Rural Highways. *Milton Harris*	May, 1941	
Discussion	Sept., Oct., 1941	Dec., 1941
Pile-Driving Formulas: Progress Report of the Committee on the Bearing Value of Pile Foundations	May, 1941	
Discussion	Sept., Oct., Nov., 1941	Dec., 1941
Wind Stress Analysis by the *K*-Percentage Method. *F. P. Witmer*	June, 1941	
Discussion	Sept., Oct., Nov., 1941	Dec., 1941
Salts in Irrigation Water. *Raymond A. Hill*	June, 1941	
Discussion	Sept., Oct., 1941	Dec., 1941
Cost of Public Services in Residential Areas. *F. Dodd McHugh*	June, 1941	
Discussion	Oct., Nov., 1941	Dec., 1941
Method of Predicting the Runoff from Rainfall. *Ray K. Linsley, Jr., and William C. Ackermann*	June, 1941	
Discussion	Oct., Nov., 1941	Dec., 1941
An Investigation of Plate Girder Web Splices. *J. M. Garrelts and I. E. Madsen*	June, 1941	
Discussion	Sept., Oct., 1941	Dec., 1941
A Direct Method of Flood Routing. *C. O. Wisler and E. F. Brater*	June, 1941	
Discussion	Oct., Nov., 1941	Dec., 1941
Land Surveys and Titles: Second Progress Report of the Joint Committee of the Real Property Division, American Bar Association, and the Surveying and Mapping Division, American Society of Civil Engineers	June, 1941	
Discussion	Oct., Nov., 1941	Dec., 1941
Design and Construction of San Gabriel Dam No. 1. *Paul Baumann*	Sept., 1941	
Discussion	Nov., 1941	Jan., 1942
Pipe-Line Flow of Solids in Suspension: A Symposium	Oct., 1941	
Discussion	Oct., 1941	Feb., 1942
Fundamental Aspects of the Depreciation Problem: A Symposium	Nov., 1941	
Discussion	Nov., 1941	Mar., 1942
Stability of Granular Materials. *R. E. Glover and F. E. Cornwell*	Nov., 1941	
Discussion	Nov., 1941	Mar., 1942

NOTE.—The closing dates herein published are final except when names of prospective discussers are registered for special extension of time.

* Publication of closing discussion pending.

CONTENTS FOR NOVEMBER, 1941

PAPERS

PAGE

Fundamental Aspects of the Depreciation Problem: A Symposium............ 1611

Stability of Granular Materials.
By R. E. Glover, Esq., and F. E. Cornwell, Esq.......................... 1639

Timber Friction Pile Foundations.
By Frank M. Masters, M. Am. Soc. C. E............................... 1657

Bending Moments in the Walls of Rectangular Tanks.
By Dana Young, Assoc. M. Am. Soc. C. E............................... 1683

Energy Loss at the Base of a Free Overfall.
By Walter L. Moore, Jun. Am. Soc. C. E............................... 1697

DISCUSSIONS

Fundamental Aspects of the Depreciation Problem: A Symposium.
By Messrs. Edwin F. Wendt and L. T. Fleming, and Anson Marston........ 1715

Analysis of Statically Indeterminate Trussed Structures by Successive Approximations.
By O. T. Voodhigula, Jun. Am. Soc. C. E............................. 1722

Stability of Granular Materials.
By R. G. Hennes, Assoc. M. Am. Soc. C. E............................. 1723

Cavitation in Outlet Conduits of High Dams.
By Harold A. Thomas, M. Am. Soc. C. E., and Emil P. Schuleen, Assoc. M. Am. Soc. C. E... 1725

Hydraulics of Sprinkling Systems for Irrigation.
By J. E. Christiansen, Assoc. M. Am. Soc. C. E....................... 1738

CONTENTS FOR NOVEMBER, 1941 (*Continued*)

PAGE

Missouri River Slope and Sediment.
By Messrs. Leo M. Odom, E. W. Lane, and Elliott J. Dent 1741

Laboratory Investigations of Soils at Flushing Meadow Park.
By Donald M. Burmister, Assoc. M. Am. Soc. C. E. 1750

Concrete in Sea Water: A Revised Viewpoint Needed.
By T. B. Rights, Assoc. M. Am. Soc. C. E. 1757

Tunnel Construction, Sixth Avenue Subway, New York, N. Y.
By John H. Myers, M. Am. Soc. C. E. 1759

Cost of Public Services in Residential Areas.
By Howard Whipple Green, M. Am. Soc. C. E. 1761

A Direct Method of Flood Routing.
By Messrs. W. B. Langbein, Fred E. Tatum, and H. C. Woster 1763

Method of Predicting the Runoff from Rainfall.
By Messrs. Richard van Vliet, and LeRoy K. Sherman 1768

Land Surveys and Titles: Second Progress Report of the Joint Committee of the Real Property Division, American Bar Association, and the Surveying and Mapping Division, American Society of Civil Engineers.
By T. J. Rodhouse, Jr., Assoc. M. Am. Soc. C. E. 1771

Wind Stress Analysis by the *K*-Percentage Method.
By Otto Gottschalk, Esq. 1777

Design and Construction of San Gabriel Dam No. 1.
By E. Soucek, Jun. Am. Soc. C. E. 1783

Pile-Driving Formulas: Progress Report of the Committee on the Bearing Value of Pile Foundations.
By Messrs. Howard T. Evans, William G. Atwood, Donald M. Burmister, Wallace E. Belcher, Clement C. Williams, and D. P. Krynine 1784

For Index to all Papers, the discussion of which is current in PROCEEDINGS, *see page 2*

The Society is not responsible for any statement made or opinion expressed in its publications

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
PAPERS

FUNDAMENTAL ASPECTS OF THE DEPRECIATION PROBLEM

A SYMPOSIUM

PAGE

Foreword.
By Maurice R. Scharff, M. Am. Soc. C. E. 1612

Relationship to Regulated Industry.
By J. A. Walls, M. Am. Soc. C. E. 1613

Relationship to Competitive Industry.
By Eugene L. Grant, M. Am. Soc. C. E. 1618

Relationship to Public Works and Government Finance.
By Roy W. Crum and Robley Winfrey, Members, Am. Soc. C. E. 1631

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **March, 1942.** Opening discussion on this paper appears elsewhere in this issue.

FOREWORD

BY MAURICE R. SCHARFF,[1] M. AM. SOC. C. E.

The Committee of the Power Division on Depreciation began its work in 1938 as a Sub-Committee of the Committee on Power Cost. It soon became apparent that the subject of depreciation was of interest to more than the Power Division, and eight Divisions became affiliated with it to create a Joint Committee on Depreciation. This Committee was created under the auspices of the Power Division, with the writer as general chairman, and includes in its membership the following Members of the Society:

C. A. Farwell (City Planning Division)
H. O. Locher (Construction Division)
Edwin F. Wendt (Engineering Economics Division)
Roy W. Crum (Highway Division), Robley Winfrey (Alternate)
E. L. Myers (Irrigation Division)
Maurice R. Scharff (Power Division)
E. B. Black (Sanitary Engineering Division)
S. S. Steinberg (Surveying and Mapping Division)
William G. Atwood (Waterways Division)

From time to time the Committee has undertaken to advance its objectives by the preparation of a number of technical papers on the depreciation problem. These have been circulated among members of the Committee; but a conviction has been growing that to advance a program of detailed, technical, and complex discussion is of little value unless the basic and simple definitions underlying it have been established previously; that is, until a common language has been outlined and an agreement has been reached as to its fundamental characteristics.

That is the central purpose of this Symposium: To avoid emphasis on technical details and make a new beginning toward formulating a preliminary agreement on fundamentals, upon which, later, can be built a sturdier "superstructure."

---

[1] Cons. Engr., New York, N. Y.

RELATIONSHIP TO REGULATED INDUSTRY

By J. A. Walls,[2] M. Am. Soc. C. E.

Synopsis

Ever since the advent of the active regulation of public utilities, the problem of depreciation has been and now is in a very confused state. In an attempt to find some satisfactory solution to the problem, a great deal of effort has been spent by independent writers, public utilities, and regulatory authorities. Numerous treatises, reports, and books have been written on the subject. Numerous concepts and methods of depreciation have been advanced, and many different solutions advocated. The present situation might appear to be a mere battle of words, or a battle over theories and definitions, which not only fails to clarify the situation but, on the contrary, obscures the basic factors that control the problem.

As a part of the Symposium on Depreciation devoted to a consideration of the fundamental aspects of the problem, this paper approaches the problem from the viewpoint of the actualities of depreciation and it is free of theoretical formulas and definitions. Two current views on the problem are examined and their relative merits are discussed. One of these views is based on the so-called age-life theory and definitions resulting therefrom that involve assumptions as to the future or remaining life of property. The other view is that depreciation cannot be made to suit a definition, but rather the definition must truly express the actualities of depreciation. This view states that depreciation in property as of any time is an accomplished fact, and, to determine that fact, engineering and technical knowledge should be used in the same way as it is employed in connection with the safety standards, replacements of property, and selections of equipment. The writer favors the latter view.

The basic purpose of this Symposium on depreciation has wisely been limited to a consideration of the general aspects of the problem with the aim of exploring the true character of depreciation and the possibility of its measurement by engineering and scientific processes. Engineers are interested as to the rôle that the profession should play in helping to determine the proper treatment of this much discussed problem.

The most difficult aspects of the depreciation problem confronting public utilities today are those in connection with estimating depreciation for accounting and with determining depreciation for rate-making purposes. These difficulties arise from certain existing facts, circumstances, and contentions, the more troublesome of which may be stated as follows:

---

[2] Pres., Pennsylvania Water & Power Co., Baltimore, Md.

(*a*) The earnings should be based upon fair return on the fair value of property devoted to public service.

(*b*) It has been suggested that depreciation taking place within any period of time is the difference in fair value at the beginning and end of the period.

(*c*) Some advocate specifically that public utility earnings should be limited to a "fair return" on the original cost less the balance in the depreciation reserve resulting from either the actual charges made or charges that should have been made in the past against operating expense on an age-life method of accounting for depreciation.

(*d*) Most enterprises, including public utilities, pass through three stages during their economic life—namely, (1) the development stage; (2) the normal productive life stage; and (3) the declining stage. The actual capacity of enterprises to make substantial provision for depreciation necessarily will often be more or less limited and difficult in the first and third stages and sometimes, by general business conditions, in the second.

(*e*) The desire is often expressed that the sums, however arrived at, that are currently set aside for depreciation and charged as operating expense should serve the purposes of accounting and rate making.

In attempts to satisfy these various facts, circumstances, and contentions, numerous definitions of depreciation have been formulated, many of which are mere combinations of words that do not contribute sufficiently toward a clarification of the problem. As a consequence, numerous "concepts" have grown up which seem to have for their purpose the conveyance of some factual or real substance to the various definitions of depreciation. One of the present problems is the formulation of a conception of depreciation that will afford a common understanding as to the true scope and character of the problem. Such a concept should recognize that depreciation cannot be made to suit a definition, but rather the definition must truly express the actualities of depreciation.

The approach to a common understanding of depreciation must, of necessity, give recognition to the manner in which, and the purposes for which, the result will be used. For example, depreciation related to reproduction cost new of a replica of an old item of property might properly include certain elements, or the effects of certain causes, that would not properly be included in depreciation related to reproduction cost of old facilities in terms of modern property; and, similarly, depreciation related to original cost may properly consist of still another group of elements.

It has been contended that depreciation at any given time cannot be measured by engineering or scientific methods with reasonable accuracy, and that future life can be estimated with greater accuracy either by deducting from a reasonable original estimate of life expectancy the elapsed life of the property, or by some other method. On the other hand, it has been claimed that the effect of forces that cause depreciation are generally measurable by engineering and scientific methods at any given time with greater accuracy and reliability than remaining life can be estimated.

If the assumptions on which the first group of concepts are founded were correct, then in so far as proponents of this group of concepts are concerned, the problem resolves into an estimate of total life (present age plus remaining life) and the application of mere arithmetical processes in the treatment of depreciation. Since the present age ordinarily may be ascertained, and since it is asserted that the necessary arithmetical processes are available, the indeterminate part of the problem simply reduces to an estimate of the remaining life. Therefore, particular interest is attached to the methods by which, and accuracy and reliability with which, such life can be estimated.

The future life of any element of property will be controlled by the combined action of those forces that cause depreciation—namely, physical (wear and tear, decay, action of the elements, etc.); and functional (inadequacy, obsolescence, changes in the art, changes in demand, and requirement of public authorities, etc.). The forces that cause functional depreciation are essentially economic in character, some of the more important of which are:

(1) The price of economically feasible new facilities, which prices will be controlled by manufacturing methods and general price level;

(2) Greater efficiency, capacity, reliability, better operating characteristics, less operating costs and hazards, or any other similar characteristics that may contribute to the justification of the replacement of facilities; and

(3) The rate of growth of the particular enterprise and of the territory or area in which products are sold or service is rendered.

Do not the same forces which control remaining life also control the amount of depreciation at any given time? However, is there not a very fundamental difference between the problem of measuring depreciation and estimating future life at a given time? In measuring the depreciation existing at a given time, comparatively reliable bases are available or obtainable as to the extent to which the forces causing depreciation have already acted. In estimating future life, however, there is little to serve as a guide as to when, and to what extent, these forces may or may not act. All estimates of future life involve uncertainties and are assumptions, and the farther these estimates are extended into the future, the less reliable they become.

The assertion is frequently heard that future life of public utility property can be estimated on an actuarial or statistical basis in the manner that insurance companies treat human lives. The correctness of such a conclusion seems open to question. Insurance companies deal with large numbers of humans whose life spans, by nature, vary comparatively slowly from generation to generation. They also segregate individuals by occupations, races, sexes, etc., into groups such that the risks may be appraised and made as nearly uniform for each group as possible. The extreme caution with which insurance companies use even their large volume of reliable statistics as a guide to future life is clearly shown by the care they use when they insure life against new and unknown risks.

The forces that control human-life characteristics are obviously not comparable to those that govern life characteristics of most utility property, and for very sound and obvious reasons. Consider, for example, the electric utility industry, which has grown from relative insignificance to its present status during the past several decades. Coincident with, and inseparably related to, this growth is the fact that the facilities that comprise the industry's property have been continually improved. Many of these improvements have resulted in functional depreciation, which has been one of the predominant causes of retirement of property in the past. Other improvements have resulted in a change in life span of facilities.

At the same time, the improvements in most property have made it impossible for the life of one generation of property to serve as a reliable guide to the life of a succeeding generation of property. Therefore, it may be true that in the case of most electric utility facilities, statistical treatment will not be possible until statistics are available of lives that have not been affected by improvements.

In the case of a relatively small proportion of the facilities of most utilities, proper statistical data may be a reliable guide to future life. However, with the possible exception of communication utilities, this proportion susceptible of statistical treatment is not a large percentage of the total property and thus does not carry much weight in the problem as a whole.

A general conclusion as to estimates of future life for most electric utility property might be stated briefly. Such estimates, even for the near future, have limited accuracy and reliability, and, when extended into the more distant future, they would appear to have even less accuracy and reliability. Even if future life could be determined with absolute accuracy, it is not, of itself, the measure of depreciation. An estimate of future life is merely an element to be considered to the extent that it may be a measure of future usefulness, but it should be recognized that future or remaining life is not necessarily synonymous with future usefulness.

The problem of measuring depreciation at a given time is fundamentally different from that of estimating future life in years. The factors that have caused depreciation are well known to engineers, for these are constantly applied by them in selections and replacements of facilities. Measurements of depreciation, like the selection and replacement of property, are essentially comparative procedures in which an item of existing property is compared in a proper manner with an available modern item of property. In such a comparison, the various points of difference such as efficiency, reliability, etc., are evaluated. Technical data and experience are available that can be utilized in making such valuations. There are also numerous technical and scientific data, tests, and procedures that are valuable aids in the determination of physical condition. An evaluation of the foregoing having been arrived at, the result must be expressed in terms appropriate to the use for which the determination is made.

The ascertainment of depreciation may well be a subject for engineering investigation and careful exercise of judgment rather than something that can be treated by those not familiar with engineering matters or something that can be solved by using perhaps the simplest arithmetic. Replacements and selections of property often require the exercise of sound engineering judgment, and decisions are not always easy to make; but such applications of judgment are not avoided through the adoption of simple methods that may lead to unsound results. The increasing complexity of modern manufacturing processes requires the development of engineering ability to solve the problems involved. Cannot this same engineering ability be devoted to the more accurate determination of the condition of equipment and depreciation? Therefore, is there a justification for the conclusion often hastily drawn that measurements of depreciation cannot, with study, be made with the needed accuracy and reliability?

There would seem to be no adequate justification for treating the matter of depreciation in a superficial manner merely for the purpose of attaining alleged simplicity. In estimating the rate at which depreciation may be expected to progress in the future, it seems proper that the estimated rate for the immediate future should be guided by the rate at which depreciation has progressed in the immediate past, modified by judgment as to the conditions and circumstances expected to prevail. This would seem a more accurate and reliable guide than reliance on estimates of future life based on assumptions and fragmentary statistics from the distant past when it is known that the prevailing conditions are not comparable therewith.

The problem that lies before engineers seems to be rather definite and is certainly pressing: First, a sound basic concept or common understanding of the true meaning and scope of depreciation must be developed; second, an orderly and basically sound procedure for measuring depreciation must be formulated that will use the vast volume of technical and scientific data and methods available, and, above all, one that will yield comparable results under comparable conditions and circumstances; and third, an orderly and basically sound procedure of deriving estimated rates of depreciation for the future from measurements of past depreciation must be formulated. None of these suggestions is intended to carry the implication that engineers must or can reduce the treatment of depreciation to a simple arithmetical process which will eliminate or even reduce the quality or amount of sound judgment required in the treatment of the problem.

RELATIONSHIP TO COMPETITIVE INDUSTRY

BY EUGENE L. GRANT,[3] M. AM. SOC. C. E.

SYNOPSIS

Writers on depreciation seem to agree on nothing except that other writers on the subject are somewhat confused. Many apparent differences of opinion on this controversial subject might be reconciled by recognition of the following two points:

1. A clear differentiation should be made between three fundamentally different depreciation concepts. These might be described briefly as the popular concept, the accounting concept, and the appraisal concept.
2. Depreciation estimates will enter in different ways into different kinds of business decisions; it is necessary to examine the specific alternatives presented in any case to see what depreciation estimates are relevant. All business decisions are between alternatives for the future; the only possible differences between any such alternatives are future differences. What is past is relevant only because of its possible aid in forecasting the future.

This paper illustrates these two points by brief discussions of the relationship of depreciation to five types of problems existing in competitive industry, as follows:

(*a*) Engineering economy studies relative to a proposed new investment;

(*b*) Engineering economy studies relative to proposed retirements of machines and structures which are still capable of rendering service;

(*c*) Establishing selling prices;

(*d*) Making appraisals based on replacement cost new; and

(*e*) Shaping public tax policy to recognize the national interest (particularly the national defense interest) in private decisions on plant expansion and plant retirement.

THREE FUNDAMENTAL DEPRECIATION CONCEPTS

A prerequisite for any discussion of depreciation principles should be a reading of the explanation, in the treatise by J. C. Bonbright entitled "Valuation of Property,"[4] of the different concepts attached to the word "depreciation." One concept involves the popular meaning of the word; two others involve special technical meanings. Failure to realize the fundamental differences between these three meanings has been responsible for much confusion in the literature of this subject. The concepts are: Decrease in value, amortized cost, and difference in value between an existing old asset and an hypothetical new asset taken as a standard of comparison. (Bonbright gives a fourth basic concept of depreciation as "impaired serviceableness." As he states, this is not a value concept at all although it has sometimes been used to confuse the issue in value determinations. "As good as new" is not the same as "as valuable as ever." Impaired serviceableness is only one of many possible reasons for fall in value.)

---

[3] Prof. of Economics of Eng., Stanford Univ., Dept. of Civ. Eng., Stanford University, Calif.

[4] "Valuation of Property," by J. C. Bonbright, McGraw-Hill Book Co., Inc., 1937. See particularly Chapter X.

Decrease in Value.—This implies that value is somehow determined at two different dates. The value at the later date subtracted from the value at the earlier date is the "depreciation" regardless of what combination of causes may have been responsible for the value change. When "depreciation" is used in everyday speech, this is the meaning usually implied; it is also implied by most dictionary definitions. It should be noted that this meaning may be applied to various concepts of value. As explained by Bonbright, the most useful concepts of value are "market value" and "value to the owner."

Amortized Cost.—This is the concept of depreciation implied by the usual accounting practice. From the viewpoint of accounting, the cost of an asset is a prepaid expense to be apportioned among the years of its life by some more or less systematic procedure. The controversial questions here are what the estimated life should be and how the apportionment should be made. It should be emphasized that it is cost, not value, which is apportioned in orthodox accounting. Although the phrase "book value" is commonly used to describe the difference between the cost of an asset and the total of the depreciation charges made to date against the asset, this difference is more accurately described as "unamortized cost." Confused reasoning may result from a literal interpretation of those classifications of accounts which define depreciation as "loss in value." Only if value is defined in what might be called a neutral sense, as any money amount that may be associated with property, can it be said that the accounts really record value.

Difference in Value Between an Existing Old Asset and an Hypothetical New Asset Taken as a Standard of Comparison.—This is the appraisal concept of depreciation. An upper limit on the value of an old asset to its owner may be determined by considering the cost of reproducing the service which the old asset is expected to perform with the most economical new asset available for performing the same service. The present cost new of this hypothetical new asset is the depreciation base. The depreciation deducted from this to determine the value of the old asset is a measure in money terms of the inferiority of the old asset to the hypothetical new one; the old asset may be inferior for a number of reasons such as shorter life expectancy, higher prospective annual disbursements to get the service performed, and lower prospective annual receipts. The writer believes that it is in this field of measurement in appraisal depreciation that engineers can make their greatest contribution to theory and practice. Such measurements, which are primarily an engineering function, can shed light on comparisons of alternative methods of depreciation accounting. Moreover, the appraisals of which they are a part often may serve as the only criteria by which decrease in value (depreciation in the popular sense) may be estimated.

Despite the frequent confusion among them, all three of these depreciation concepts are useful. Any suggestion that two of them be given different names would doubtless be futile. Throughout the remainder of this paper they are called "popular depreciation," "accounting depreciation," and "appraisal depreciation," respectively. Some fundamental differences among these three concepts are revealed in Example 1.

Example 1.—A man bought a new home in 1932 for $6,000. In 1940 he sold it for $6,000. In the intervening years he rented it to a tenant for part of the time; and in reporting his taxable income for this rental period to the government, he made an annual depreciation deduction of $125. This was based on a $1,000 lot on which no depreciation was charged and a $5,000 house on which straight-line depreciation was calculated over an estimated 40-yr life. The purchaser arrived at his $6,000 bid by estimating the cost of buying a similar lot at $1,000 and the cost of building a comparable house in 1940 at $6,500; from this total of $7,500 he deducted $1,500 because of the shorter life expectancy, higher early maintenance costs, and less modern design of an 8-yr-old house.

In this incident the popular depreciation was evidently zero, as the market value in 1940 was the same as in 1932. The accounting depreciation was $1,000—that is, $\frac{8}{40}$ of $5,000. The appraisal depreciation was $1,500.

Increase in residence construction costs between 1932 and 1940 had compensated for the causes making for decline in value so that there was no popular depreciation. Changes in price levels had no effect on accounting depreciation, as the accepted principles of accounting call for basing depreciation charges on cost. Price level changes did not affect appraisal depreciation for quite a different reason; they had already been taken into account by using the present cost of a new asset as the depreciation base.

ENGINEERING ECONOMY STUDIES RELATIVE TO A PROPOSED NEW INVESTMENT

Answering the question "Will it pay?" with respect to a proposed plant investment may involve the conversion of a first cost, P, an estimated life, n, and a salvage value, L, to an equivalent uniform annual cost. This requires the use of an interest rate, i, which should be the minimum rate of return that will make the investment attractive. This minimum attractive return should depend on both the prospective return obtainable from other investments and the apparent risk of the particular investment in question.

With zero salvage value, the equivalent uniform annual cost is obtained by multiplying the first cost by the capital recovery factor, $\frac{i\,(1+i)^n}{(1+i)^n - 1}$. With a salvage value, the annual cost of capital recovery is

$$C = (P - L)\,\frac{i\,(1+i)^n}{(1+i)^n - 1} + L\,i \qquad \text{(1)}$$

It may be shown readily that this is equal to the sum of interest on first cost and a "sinking-fund depreciation" annuity, provided the interest rate used on first cost is also used for the sinking fund. Thus the annual cost of capital recovery may be described as interest plus depreciation.

As the two preceding paragraphs describe conventional theory in engineering economy, they require no detailed explanation. However, a point not always understood should be emphasized—namely, that this equivalence calculation simply states: "Here is a uniform series of payments which, with the final salvage value, would serve to pay back the first cost in n years with the minimum attractive rate of return i." The calculation is not concerned with the central problem of depreciation accounting, the allotting of cost among a number of different years. The equivalence holds good regardless of what plan of accounting is to be used and has no relation to any actual or hypothetical sinking fund for the replacement of invested capital.

Although such a capital recovery calculation may be thought of as the amortization of cost with interest and thus appears to combine a recognition of the time value of money with the viewpoint of accounting depreciation, it is really the viewpoint of popular depreciation—prospective fall in value regardless of cause—which is relevant. For this reason, estimates of life and salvage value in economy studies to determine whether a proposed plant investment will pay may properly differ from the estimates of life and salvage value to be used in the depreciation accounting if the investment actually is made.

Example 2.—For instance, imagine that the home owner in Example 1 had anticipated that a house purchased in 1932 would be salable a number of years later for the same price. Obviously this forecast of no popular depreciation would have been relevant in an economy study comparing home ownership and renting. If price level changes are forecast as likely to influence possible resale value of a property, this forecast should influence the annual cost of capital recovery used in a study to determine the economy of acquiring that property. However, the forecast of price level changes has no place in depreciation accounting as it is ruled out by the conventions of accounting.

More important than this in many economy studies is the forecast that long before the date of its retirement the value to the owner of a proposed machine or plant will be very low. This forecast may be based on past experience that improvements in design render plant obsolete in a short time, or on an expectation of the termination of the present demand for the product, or on other reasoning. It is a common policy, particularly in the manufacturing industries, to require that new machinery "pay for itself in three years" or in some other short time from one to five years. It is also common for the machine, installed on the basis that it must pay for itself in three years, to have its depreciation accounting based on a 10-yr life. Past experience might show that similar machines had served fifteen years before retirement. Recent regulations of the Bureau of Internal Revenue, U. S. Department of the Treasury, have had the effect of bringing the accounting depreciation—at least for income tax purposes—more closely into line with past evidence of total realized life, and have thus increased the discrepancy between the life assumed in the economy study and that assumed in the subsequent depreciation accounting.

ECONOMY STUDIES FOR PROPOSED RETIREMENTS

Perhaps most property retirements in modern industry are retirements of machines and structures which are still capable of rendering service. Such retirements are made because of some one's decision that it pays to make them. Unless such decisions are made merely on a "hunch" basis, they require an estimate of the cost of extending the service of the old asset for some specified future time.

It should be obvious that this "cost" is a money difference between disposing of the asset at once and disposing of it at the specified future date. The only "depreciation" involved is the difference between the net amount realizable from the immediate sale of the asset and the net amount realizable from its future sale. This is depreciation in the popular sense of decrease in value; the relevant "value" is net price in a second-hand or scrap market, or value to the owner for some other purpose. Accounting depreciation based on the past cost of the asset is irrelevant; so also is the difference between the unamortized cost ("book value") and the present net realizable value. The money originally spent for the asset has been spent regardless of what decision is now made about continuing it in service.

Although this should be obvious, unfortunatey it is not. The literature of replacement economy is full of formulas which, as part of the attempt to determine whether a proposed replacement will pay, add to the first cost of the proposed new asset the "loss" which the books would record on the disposal of the old asset—that is, the difference between the "book value" and the net realizable value. (Although under individual unit depreciation accounting, if the "book value" of a retired asset is greater than its net realizable market value, the difference must be written off on the books [either as a charge to current expense or in unusual cases as a charge directly to surplus]; the usual description of this difference as a "loss" which occurs at the time of the retirement is misleading. The so-called "loss" is simply a portion of a prepaid expense which would have been allotted against past accounting periods if correct forecasts of life and salvage value had been made; because it is impracticable to go back and correct previous accounting statements, this must be charged off at the time of the retirement. Under group depreciation accounting, no such "loss" entry is made as, presumably, a retirement short of the estimated average life of a group of assets will be balanced by another at more than the average life. This viewpoint is implied in recent regulations of the Bureau of Internal Revenue; "losses" on retirements made sooner than the estimated average life are no longer allowable as tax deductions except in extraordinary circumstances.)

The most celebrated of these formulas[5] was first published in 1923 and has been copied year after year since that time in textbooks and handbooks; it still (1941) seems to be "going strong" after eighteen years. One proposed formula[6] went so far as to charge against the proposed new asset not only the "loss" on the current retirement but also a portion of the "losses" on previous retirements of a series of predecessor assets which performed the same service!

---

[5] "Formulas for Computing Economics of Labor-Saving Equipment," *Mechanical Engineering*, Vol. 45, September, 1923, p. 564.

It is the writer's impression that confused reasoning on this matter is not so common in industry as would be suggested by the high percentage of the replacement formulas which contain this mistake. Nevertheless, it is common enough; there is no question that high book value of old assets often operates as a deterrent to economical retirement. The writer believes that this error is more common in large concerns than in small ones. The control mechanism of a large corporation often depends on accounting figures to such an extent that its employees impart a fictitious reality to the apparent evidence of the accounts. The owner-manager of a small concern is likely to be more conscious of his competitive position and of the actual differences between alternatives.

Modern writers on semantics have emphasized how readily one may be misled by words or phrases that have unfavorable associations. Two such phrases may be partly responsible for some of the confusion that exists about depreciation. One is the definition of accounting depreciation as "loss" in value; the other is the common description of unamortized cost as "unrecovered investment." "Loss" has an unpleasant sound which somehow makes depreciation seem worse than other operating expenses. "Unrecovered investment," despite its sanction by the Bureau of Internal Revenue, is an inaccurate description of "unamortized cost." In some cases the "investment" in a machine actually may be "recovered" in its first few months of service, and in other cases it may never be recovered; neither fact has any relation to the depreciation that has been recorded on the books. (Even if "book value" really were unrecovered investment, it would not be relevant in a replacement economy study.)

Depreciation in Pricing Policy

In a competitive market, prices are a result of the bargaining of possible buyers and possible sellers. Depreciation as one element of cost enters into the bidding of the possible sellers. According to economic theory, under free competition prices tend to reflect costs to the marginal producer—that is, the highest cost producer who is actually able to make a sale. The supply side of the market depends on decisions by individual possible sellers as to whether or not they will sell at some given price. In practice, the "cost" to any possible seller which should be relevant in his decision whether or not to sell at any given price depends on whether or not he has the necessary plant to produce the product. Thus two types of decisions involving depreciation enter into the bidding of possible sellers.

One is a decision whether prospective prices are high enough to justify the provision of new plant facilities. As already pointed out, in many competitive industries new construction is not undertaken unless there is the prospect of capital recovery within a much shorter period than the estimated total service life of the plant. A price higher than production cost by the most economical new method, including in production cost a much larger depreciation allowance than will subsequently be included in the cost accounts, tends to attract new capacity into the market. In this manner, improvements in design which make it possible to produce more economically tend to reduce the return to the owners of older, less economical plant.

---

[6] "Economic Life of Equipment," by H. O. Vorlander and F. E. Raymond, *Transactions*, A.S.M.E., Vol. 54 (1932), RP–54–2.

The other decision is whether or not a given market price justifies continuing to operate an existing plant. If the owners of an old plant are faced with a choice between the alternatives (*a*) to sell at slightly more than the increment cost of production and (*b*) to shut down the plant, it is to their advantage to choose alternative (*a*). The only depreciation properly included in this increment cost is the difference between the decrease in value, that is, depreciation in the popular sense, which will take place if the plant is operated and if it is shut down. In the common instances in which decrease in value is practically independent of operation, this increment of depreciation entering into pricing policy will be practically nothing.

The danger that excess productive capacity in an industry may result in "cut-throat" competition in which prices are forced down to increment cost of production has its influence on economy studies relative to proposed new capacity. The "pay-for-itself-in-three-years" requirement is often partly attributable to this hazard.

Under the imperfect competition that actually exists in most competitive industries, prices in an excess capacity industry may not actually fall to the increment cost to the marginal producer. Producers may prefer a partial shutdown rather than a sale close to increment cost of production. If total allocated book cost determines the point at which this shutdown occurs, it is accounting depreciation which actually is relevant. It should be emphasized that it is what each prospective seller thinks at the time that his relevant costs are, not what subsequent events prove them to have been, that determines his pricing policy.

Because depreciation may enter into competitive prices in such different ways, it is desirable that contracts in which the determination of price involves the determination of "cost" should not depend on a competitive price analogy for the establishment of depreciation cost. Such contracts should provide definite money amounts for depreciation (or a means of readily determining the money amounts) and thus avoid the necessity for adjudicating disputes which inherently are extremely difficult to adjudicate.

Depreciation in Appraisals Based on Replacement Cost

An appraisal of an asset on the basis of its replacement cost should give an answer to the question, "What price for this asset would provide its future services at the same annual cost as would be obtainable from the most economical new asset now available to reproduce the service?" The depreciation deduction from the price of the hypothetical new asset should be a money measure of the inferiority of the existing old asset to the most economical new one. The old asset is usually (although not always) inferior in having shorter life expectancy; it may also be inferior in requiring greater annual disbursements to obtain the service performed or in having the prospect of lower annual receipts.

The theory that should govern this depreciation measurement is the same theory that applies to a somewhat comparable problem in engineering economy—that of determining the investment justified to secure an increase in life or a decrease in annual disbursements. Consider, for instance, the following problem.

Example 3.—A structure, A, has a first cost of $1,000, a prospective life of ten years, and involves estimated annual disbursements of $500. An alternative structure, B, has a prospective life of twenty years with estimated annual disbursements of $500. With interest at 7%, what is the maximum amount that could be paid for B without realizing higher annual costs than A?

This problem may be solved by letting x = the first cost of B which would just make annual costs "break even." Each investment may be converted into an equivalent annual cost by multiplying by the appropriate capital recovery factor. Equating the annual costs for A and B: $1,000 (0.14238) + $500 = x (0.09439) + $500. That is, x = $1,508.

Because of the time value of money, in this problem a 100% increase in life justifies only a 50% increase in investment. Only with 0% interest would it pay to spend twice as much for a 20-yr service as for a 10-yr service. The interest rate used in this calculation should be the minimum attractive rate of return on investment, considering the risk and the return available from other investments. Obviously this return should be greater than the bare cost of borrowed money. The higher the minimum attractive return, the lower the valuation placed on additional life expectancy.

A prospective saving in annual disbursements may justify a much higher investment. For instance, if structure B had the added advantage of annual disbursements of $300 instead of $500, the equation would be: $1,000 (0.14238) + $500 = x (0.09439) + $300; or x = $3,627.

Example 4.—In applying this viewpoint to the appraisal of existing old assets by comparing them to hypothetical new ones, it is the value of the shorter-lived asset that is the unknown in the equation. For example, the most economical new asset that might be used to perform the service of an existing old one has a first cost of $1,000, an estimated life of twenty years, and estimated annual disbursements of $500. The old asset has a remaining life expectancy of ten years and estimated annual disbursements of $500. With interest at 7%, how much could be paid for the old asset to just "break even" on the annual costs with the new one?

Let x = the value of the old asset to "break even"; x (0.14238) + $500 = $1,000 (0.09439) + $500; and x = $663.

If the new asset involves lower prospective annual disbursements, this decreases the value of the old asset. For instance, if the annual disbursements with the new asset are $450 instead of $500, x (0.14238) + $500 = $1,000 (0.09439) + $450; and x = $312.

If the annual disbursements for the new asset are $405 or less, the equation shows that the old asset has no value in comparison with the new one. That is, on the basis of a 20-yr life for the new asset it would pay to make an immediate replacement.

The preceding paragraphs explain a theory on which depreciation estimates in replacement cost appraisals might reasonably be made. The writer believes that so-called reproduction-cost appraisals in competitive industry have little meaning unless they are really based on the cost of replacing the service. He realizes, of course, that few appraisals are actually made in this manner.

The elements in such an equal-annual-cost appraisal are: (1) The expected service to be performed; (2) the selection of the hypothetical new asset to perform it and the estimate of its cost; (3) the estimation of the remaining life expectancy of the old asset; (4) the estimation of the life expectancy of the new asset; (5) the selection of an interest rate or minimum attractive return; and (6) the estimation of the saving in annual disbursements or the increase in annual receipts to be expected from the new asset, or both.

Each of these elements deserves a brief comment:

(1) The service should be the one that is to be performed in the future by the old asset. The result of the appraisal must be recognized as relevant only to the value, to the owner, of the asset for this particular service. For some other service a different conclusion regarding value might be reached.

(2) The reason for the relevance of replacement cost in an appraisal is its relation to the cost of replacing a service. This implies that the replacement cost used should be that of a new asset which would provide the contemplated service most economically. If there have been improvements in design or changes in the service performed, the new asset will not be physically identical with the old one; often it will be entirely different.

(3) The estimate of remaining life expectancy does not necessarily depend on age. As stated at the beginning of this paper, the only possible differences between any two alternatives for the future are future differences. It is the look forward that is relevant in the appraisal; the look backward (at age) is only useful if it helps in the look forward (at life expectancy). Sometimes—if the only event limiting the life of either an old or new asset is the prospective termination of the service—the life expectancy of the old asset may be the same as that of any new one that might replace it. In such cases, if there is no difference in prospective annual disbursements or receipts, the old asset has no value-inferiority to the hypothetical new one and the appraisal depreciation is zero. Sometimes it may seem likely that it will pay to retire the old asset very soon. It is the prospective date when it would pay to retire the asset from this particular service that is relevant in estimating its remaining life, rather than the prospective date of termination of its physical life. Where the estimate is being made from a mortality curve, the curve will usually indicate the remaining life expectancy of an old asset to be greater than the difference between the average life indicated by the curve and the present age of the asset.

(4) If a mortality curve based on past experience is used as a basis for the estimate, the average life indicated by the curve should appropriately be used as the expected life of the hypothetical new asset. On the other hand, if the prospect of obsolescence of a new asset is so great that an economy study rela-

tive to its proposed installation might reasonably be based on the requirement that it must "pay for itself" in a short time, it would seem appropriate to use that same short time as the expected life of the hypothetical new asset.

(5) The recognition of the time value of money is simply the recognition that an extra investment to secure longer life or lower operating disbursements or increased receipts is not justified without the prospect of a return. The interest rate appropriate to use should be the minimum attractive return which will justify an investment considering the risk. Usually this will be higher than either the cost of borrowed money or the interest rate obtainable on a conservatively invested sinking fund.

(6) It is the prospective saving in annual disbursements from the most economical replacement asset that is often given inadequate consideration in commercial appraisals. This saving may result from a combination of causes which might be described as wearing out, obsolescence (or obsoleteness), and inadequacy. Even if the new asset selected as a standard of comparison is identical with the old asset, the new asset is likely to require lower annual operation and maintenance expenditures because of its newness and the lack of wear and tear. If the new asset is superior because of improvements in design, it will have lower annual disbursements for operation. If the service to be performed has changed since the old asset was installed, a new asset will be better adapted to the present service and will cost less to operate. If all of these influences are present, it should be emphasized that they combine to create a value-inferiority of the old asset to the hypothetical new one.

An apparent paradox here is that the less the contempleted service, the less will be this value-inferiority of an existing old asset to a hypothetical new one. If an old plant is to be used merely for stand-by purposes, no saving at all in annual disbursements might be possible from the substitution of a new plant. However, in this connection it should be emphasized that the appropriate substitute plant for stand-by service may be quite different from that for regular operation; the price of similar machinery in the second-hand market might provide the relevant "replacement cost" from which no depreciation deduction would need to be made. It is also true that the alternative of abandoning the stand-by service may place an upper limit on the value of that service.

Subject to certain very important limitations, the preceding discussion of appraisal depreciation on an equal-annual-cost basis may be described as an advocacy of sinking-fund depreciation in appraisals—or present worth depreciation as it is somewhat more accurately called.[7] These important limitations are:

(*a*) An interest rate equal to the minimum attractive return sufficient to justify an extra investment is usually higher than the rate commonly used in sinking-fund depreciation. This tends toward higher valuations than usually are obtained by the sinking-fund method.

(*b*) A recognition that it is the comparison of the remaining life expectancy of the old asset with the life expectancy of the new asset will often tend to give higher valuations than the sinking-fund method based on the present age of the old asset.

---

[7] "Engineering Valuation," by Anson Marston and T. R. Agg, 1st Ed., 1936, McGraw-Hill Book Co., Inc., New York, N. Y.

(*c*) The use of the cost of the most economical new plant to provide the service as the depreciation base rather than the cost of reproduction of a plant identical with the old one, and consideration of the prospective saving in annual disbursements from such a plant, will generally tend toward much lower valuations than will be found in conventional appraisals by the sinking-fund method.

The writer believes that in competitive industry this last limitation is likely to be the most important one, and that appraisals made on a true equal annual cost basis are likely to give values that not only are below reproduction cost appraisals on a sinking-fund basis but also are below such appraisals on a straight-line basis.

A word of caution is necessary regarding the addition of the appraised values of separate machines and structures in a manufacturing plant to reach an appraised value for the plant as a whole. This will often give an overvaluation; consideration of the alternative of replacing the plant as a whole with one of quite different design may give a much lower equal-annual-cost value for the entire plant than consideration of separate alternatives of replacing parts of the plant without changing its basic design or arrangement.

The National Interest in the Effect of Depreciation Tax Policy on Private Decisions Regarding Plant Expansion and Retirement

Defense needs have made every one conscious that there is a national interest in the nation having the best possible plant for the production of goods and services. This means the most economical plant for the bulk of production, and stand-by capacity of less economical plant to meet emergency needs. Tax policy with regard to depreciation may have two undesirable effects: It may handicap plant expansion and the substitution of more economical plant for less economical plant; and it may make it pay to retire plant which might subsequently be useful.

If the decrease in value of an asset is much greater than the allowable tax deduction for depreciation, what appears for tax purposes to be income may actually be recovered capital. Income and excess profits taxes may take as much as 62% of this recovered capital. The prospect of this can, and does, operate as a deterrent to plant expansion. This fact was given belated recognition in the excess profits law which permitted amortization of new plant for tax purposes in five years when it was certified by the military services to be necessary for defense needs.

Property taxes may make it pay to retire plant that would otherwise be kept to permit expansion of production during emergencies. The policy of basing allowable depreciation rates for tax purposes on the evidence of actual retirements may have a similar effect.

There is serious need for a study of the effect of depreciation tax policy on business decisions in private industry.

A Suggestion Regarding Depreciation Accounting

Many of the matters covered in this paper do not bear on the question of choice of the best of the various possible methods of making the time distribution, on the books of account, of the prepaid expense of fixed assets. This paper, rather, is concentrated on a discussion of many depreciation problems in which no scheme of depreciation accounting has any relevance. Nevertheless certain points brought out in this paper bear definitely on the selection of a depreciation accounting policy. These points may be summarized as follows:

1. In an industry in which production methods are rapidly improving, appraisals based on the cost of reproducing the service may show that fixed assets have a very low value to their owners within a short time after their installation even though it may be many years before their probable retirement.
2. The prospect that this will occur is commonly recognized in competitive industry by a refusal to purchase new fixed assets unless it seems likely that they will "pay for themselves" in a much shorter time than will be used as their lives in subsequent depreciation accounting.
3. Because of this, the limit on competitive prices which exists due to the possibility of production at the "cost" obtainable in a new plant of the most economical design practically depends on a "cost" which includes a greater depreciation allowance than is commonly included in depreciation accounting.
4. In contrast to the influence on competitive prices of this high prospective depreciation of a proposed new plant, the influence of depreciation of an old plant on competitive prices is likely to be relatively small.
5. High book values for old assets operate to discourage (irrationally, to be sure) replacements that are really economical.
6. Low depreciation rates allowable in income tax returns for new assets operate as an obstacle (an entirely rational one) to the installation of new plant.

The merits of alternative possible depreciation accounting methods may be compared by various standards. There is no inherently correct way to apportion depreciation on the books of account; it all depends on the objective being sought.

Maurice Scharff,[8] M. Am. Soc. C. E., has suggested that accounting depreciation in regulated industry be made consistent with decrease in value—that the accounting depreciation assigned to any period be based on the difference between the appraised values at the beginning and end of the period. In competitive industry the irregular operation of the forces influencing value changes would doubtless result in this method showing erratic fluctuations in annual depreciation. As some of the uses of accounting make it desirable that there be reasonable regularity in the annual depreciation charge, and that this charge be not subject to the suspicion of manipulation which would surely exist if appraisals were controlled by the management, this suggestion has obvious disadvantages.

Nevertheless it has much merit as a basis for judging the adequacy of the depreciation charges made under some regular systematic scheme of depreciation accounting. If the "decrease in value" shown by an appraisal based on the cost of reproducing the service with the most economical modern plant should be much greater than the accounting depreciation, one might reasonably infer that the accounting depreciation was inadequate.

---

[8] *Transactions*, Am. Soc. C. E., Vol. 104 (1939), p. 1132; also "Public Utility Depreciation," by Maurice Scharff, *Columbia Law Review*, June, 1938.

It is the writer's guess that engineering studies in the manufacturing industries of the value-inferiority of the existing old plant to the most economical new plant would indicate that the decrease in value in the early years of life of a plant was much greater than the accounting depreciation in most of the cases where straight-line depreciation accounting is based on actual retirement practice. This guess is consistent with the usual practice of manufacturers of requiring the likelihood of a short capital recovery period as a prerequisite to the installation of new machinery.

Changes in depreciation accounting practice in the United States have seemed to follow criticisms of current methods made on theoretical grounds. As writers on accounting theory began to point out that the distinction between a capital and a revenue expenditure was one of degree rather than of fundamental character, depreciation accounting began to develop. When the arbitrary and irregular depreciation charges made by some corporations were criticized, a trend developed toward the adoption in each industry of standard rates for each class of assets used in the industry. A current theoretical criticism is that rates are not based on actual retirement practices, and the suggestion is made that they should be based on the available evidence of actual realized lives such as might be obtained by mortality studies using the annual rate method or turnover method. Recent policies of the Bureau of Internal Revenue with regard to depreciation deduction in income tax returns are based on the viewpoint involved in this criticism. The effect of these policies has been to make substantial reductions in the annual depreciation expense shown on the books of many taxpayers, particularly those in the manufacturing industries.

A first impression might be that, because this is a trend to substitute depreciation rates based on the facts of realized lives of property for arbitrarily guessed higher rates, it is desirable. The writer holds a contrary view, particularly with respect to the manufacturing industries. That is, he believes that the decrease in value in the early years of life of much manufacturing equipment is so great that straight-line depreciation based on actual retirement experience is inadequate in those years. A more realistic change in current depreciation accounting policies would be to use a high rate in the early years of life of an asset followed by a lower rate in later years. This would bring depreciation accounting more closely into line with decrease in value, and would bring the accounting estimate of profit more closely into correspondence with the estimates made in the economy study on which the plant investment was justified in the first place.

Factual studies in various industries by engineers whose familiarity with the most economical production methods now available enabled them to estimate the value-inferiority (that is, the appraisal depreciation) of existing plants would permit determination of the past decrease in value—the popular depreciation. Such studies would permit judgments on the adequacy of the accounting depreciation which would serve to confirm or disprove the writer's present views.

RELATIONSHIP TO PUBLIC WORKS AND GOVERNMENT FINANCE

By Roy W. Crum[9] and Robley Winfrey,[10] Members, Am. Soc. C. E.

Synopsis

Physical properties depreciate regardless of whether they are privately or publicly owned or whether operated for a profit or solely for nonpecuniary purposes. It cannot be escaped, but generally public accounting methods and policies have given it no attention. Accounting for depreciation and the use of depreciation estimates in management are the possible points on which procedures involving public works and government finance may differ from those followed in privately owned industries and business. The object of this paper is to illustrate how depreciation should be used in the management of public works and enterprises operated by governments.

The administration and management of publicly owned utilities, which are supported entirely by user charges, such as water works and light and power plants, should involve in general the same principles and financial procedures as privately owned properties, and will not be discussed herein. It is true, however, that many of them are not operated under full adherence to these principles.

In the management of certain other publicly owned and operated services, of which the highway systems are perhaps the most prominent example, depreciation of the various property units is an important economic factor; but it is one which bears quite a different emphasis than in the case first cited.

In this instance knowledge about the rate at which the property units must be replaced is important to the administrator in order to enable him to plan future development as well as financial programs. The emphasis herein is upon planning for the future and upon the development of sound and adequate financial and economic programs.

Introduction

Typical of the accounting practices of most highway departments in the past is the example of an expensive piece of road machinery purchased out of current funds, and with its entire cost charged to the current year's maintenance expense, although it must continue in use for many years. Such a practice undoubtedly accounts for the money, which may be all that the law

---

[9] Director, Highway Research Board, National Research Council, Washington, D. C.

[10] Research Associate Prof., Civ. Eng., Eng. Experiment Station, Iowa State College, Ames, Iowa.

requires, but it does not give the administrative official a true picture of his operations nor any help in planning to have the necessary funds available when it becomes necessary to replace the machine.

For this reason, responsible highway officials have been giving much study during the past several years to developing systems that will enable them to account for the depreciation of the highway and other property in their charge. The same situations exist in other types of governmental service. The writers are relying principally on the highway illustration of their thought, since that is the business with which they are familiar, but they emphasize that the illustration is applicable to other activities of government.

Funds for the operation of such public services are secured through various tax imposts and direct charges to users; and in making his recommendations for financial support, the administrator is not concerned with making a return upon the investment but with providing adequate service in accordance with the wishes of the voters.

It is a fact that public accounting for governmental service does not usually distinguish between running expenses and capital investment, but nevertheless some breakdown into cost factors is necessary for the highway planner. Depreciation is the most significant of these.

It is desirable that public policy include some recognition of the depreciation of public property as a means of effecting better financial management. Knowledge of the length of service lives and of salvage values, which are the basis of the depreciation rate, could be used in the management of public services in many ways. A few of these are as follows:

(1) By application of the probable rates of retirement of property to the existing property, it is possible to estimate the retirements to be made in future years.

(2) The replacement program, estimated as in item (1), may be combined with a program of extension and betterment and may be used to formulate a financial program for the future.

(3) True annual cost can be shown only when the expense of depreciation is included. Since annual cost is the basic unit used by engineers in determining economic justification of selection of projects, as well as total revenue needed, depreciation is an important element on which complete knowledge should be had.

(4) Correct depreciation rates are necessary when making comparisons of highway costs with other modes of transportation.

In the case of highways the correct rates of depreciation to use can be determined by study and analysis of past experience. This work, along with other planning studies, is now (January, 1941) being done by forty-six state highway departments in cooperation with the U. S. Public Roads Administration. As reports of these studies become available, reliable information will be at hand on which to base estimates of depreciation and forecasts of probable future retirements.

VALUE AND DEPRECIATION

Although length of life and not money value is the factor of greater importance in operating a publicly owned business composed of thousands of items of physical property, it is necessary to take account of the property items by ordinary bookkeeping methods. On that account the public official must have a clear understanding of the nature of value and depreciation and of depreciation estimating and accounting.

Definition of Value.—Once attention is called to the point, it is quite easy to understand that it is only the anticipated worth of future services that gives value to any property. Past services and earnings are just so much "water gone by the mill." Every estimate of present value in reality is a forecast of future happenings.

The fundamental basis of value is present worth of probable future service. The fundamental basis of the fair value of each specific property item is the present worth to its owner of the probable future services to be rendered by the item during its future service life.

Manifestly, neither the number of future years of service nor the yearly return for services can be forecast with certainty or exactness; nevertheless, every sale of property and every estimate of value are based more or less consciously on the present worth of such anticipated future services as judgment indicates to be most probable. Reliable estimates of the values of industrial property units can be made only on the basis of the sound judgments of valuators, well qualified by training and experience, who give careful consideration and such weights as are just and right to every factor in each case.

Depreciation is negative value, to be determined at every date in the service life of a property unit by the sound judgment of well-qualified, impartial experts as to its probable future service life and its probable future economic efficiency.

Definition of Depreciation.—The depreciation of any physical property unit, at any service age, is its loss of value, since its service started, due to decrease in the present worth of its probable future earned operation returns, below what such present worth would be if the unit were still new.

Thus, depreciation is just a question of values at different dates based on forecasted (probable) numbers of years of future service and probable values of the yearly services. Such forecasts can be made correctly only by the sound judgments of impartial persons who are qualified to understand the mortality and other technical characteristics of different kinds of physical property units, and who personally examine the particular units whose depreciations are being determined.

Property units should be reexamined by qualified persons, from time to time throughout their service lives, and the estimates of the depreciations at different dates should be kept constantly in accord with sound current forecasts of the numbers of years of future service and of the yearly operation returns for such service.

DISTINCTION BETWEEN DEPRECIATION AND MAINTENANCE

Maintenance expenses are those for keeping the property units in good condition for use. Examples are the costs of inspecting, painting, and repairing structures and equipment. Maintenance expenses are paid from yearly income, and constitute part of the yearly operation costs. Maintenance cost accountancy, therefore, consists simply in making properly classified accounting records of the yearly maintenance expenditures.

Depreciation losses of value, on the other hand, are something which cannot be made good in a physical way until new units are substituted for the depreciated units at the ends of their service lives. Depreciation expense accountancy, therefore, is a comparatively complicated process by which annual depreciation expense allowances are made each year out of income and used (usually reinvestment in new property units for replacements and additions) to create a depreciation reserve, which should be at all times just equal to the total accrued actual depreciations of all existing property units. In this way the correct annual cost is known and the depreciation reserve account shows the true loss in value of existing property, from which, of course, the present true value of existing property is determined.

USE OF DEPRECIATION IN PUBLIC ACCOUNTING

Because of the time-honored custom in governmental accounting of staying strictly in the cash accounts, depreciation of public properties simply has not been accounted for. Even for municipalities it is probably safe to state that only in recent years has there been any attempt to account for depreciation of their electric, water, and other utilities. In those few cases where cities have made allowances for depreciation in their accounting, there has been little attempt to determine accurate rates.

When, therefore, mention is made of the depreciation of other public properties, particularly those which do not produce direct revenues, immediate objection may be raised on the ground that the government is not in business for profit, is not compelled to operate its functions and businesses solely from user revenues, and that there is no need of knowing the relations between investment, expenditures, annual costs, and revenues. Nevertheless, if economy in government is to be attained, sound principles of business management must be followed. As governments more and more take over and enter into operations that might be regarded as business enterprises, either competitive or noncompetitive with private interests, it is essential to adopt accounting practices, equal to those that the government prescribes for regulated utilities. This will include the accounting for physical properties, depreciation, operating revenues, operating expenses, and the compilation of balance sheets and operating statements.

In many government-operated functions, such as printing shops, warehouses, central purchasing bureaus, state garages, and municipal utilities (not all, of course), it is probable that if all the costs were collected and reported it could be shown that such activities handled by the government could be purchased at less cost from private companies. Where there is, and where there is not, legitimate opportunity for the government in business can best be determined by accurate cost analysis and scientific evaluation of the worth of the services. Depreciation is a large item of cost in the management of any business that requires large investments in buildings and equipment; yet it is the one that government agents and the public alike are prone to forget.

Engineers in government service are responsible for the operation of huge public works. Their future and the future of public works depend upon the showing they make. Yet generally they are deprived of the opportunity of making an actual financial accounting of their enterprises because of laws prescribing archaic accounting methods or because those in charge of government accounts permit nothing but "cash and carry" accounting methods. The engineer can do much to improve the accounting of governmental activities, particularly for those which involve his works.

Accounting for Depreciation of Highways

Within the length of this paper there is opportunity to present just a suggestion of what might be done for a state highway system—in fact, what has been done, because the State of Kansas in 1941 completed a set of property investment and depreciation ledgers for its primary system of about 9,200 miles.

Obviously, before depreciation can be estimated it is necessary to account for the physical properties that depreciate. For highways, as well as for other properties, it is necessary to adopt accounting units that will permit accounting for the properties by location and kind. If any system of highways is laid out into a number of route sections with termini at more or less fixed geographical landmarks, a number of basic units are obtained, complex though they be, which will serve for accounting operations as well as for engineering and construction operations. These route sections should be governed in length by municipal, county, and state boundaries, route intersections, stream crossings, abrupt changes in topography, character of traffic, and convenient construction project termini.

Within the route section it is necessary to provide for investment accounts on the basis of the function and character of the construction, with due regard to the retirement characteristics and depreciation rates. Suggested classifications are those construction accounts tentatively recommended by the Committee on Accounting of the American Association of State Highway Officials. This classification results in the following investment accounts for each route section:

Right of way
Engineering
Roadway and drainage grading and earthwork
Drainage structures and roadway earthwork protective structures
Roadway surface and base (by roadway surface type)
Improved shoulders and approach surfacing
Bridges, viaducts, and tunnels (by individual structure)
Traffic and pedestrian services
Roadside development

The accounting unit, as will be observed from the foregoing classification, is best chosen as the dollar, with a control in miles or square yards for roadway surfacing, cubic yards for earthwork, and other suitable construction units for other items. It is difficult to identify most of the physical parts of the highway, so the average costs and the "first-in, first-out" theory must be used in accounting for certain retirements. Supporting the accounts of each route section should be construction cost reports made up from the final payment estimate and engineer's final report.

The process of estimating depreciation for the highway system need be no different from that used for other types of properties. Perhaps there is not available as much factual information and well-informed opinion as to depreciation and the factors causing depreciation of highways as there are for certain types of industrial properties, but the same basic principles and methods should be used. As the highway systems are rebuilt, expanded, and relocated, data become available that can be analyzed by accepted statistical methods to determine probable average lives and life expectancies. Again, these methods are the same as are being used by many utilities and other industries.

To account for depreciation of public properties on a correct basis is new; depreciation reserve accounts and depreciation expense in public highway accountancy until recently were wholly foreign elements. Because of a diversity of ages, qualities of construction, service conditions, types of materials, and variety of design, highway property lends itself to a modified form of group depreciation accountancy. If each route section is considered separately, all the variables can be investigated and the annual depreciation calculated for each investment account for each route section. With this breakdown, total depreciation can then be obtained for any complete route, area, and account.

To complete the accounting picture, maintenance costs, administrative costs, and other items of annual cost should be allocated to the same route sections so that the true annual cost may be obtained for comparison of types of construction, of the effects of volume and character of traffic, and for comparison of annual costs with annual revenue.

SUMMARY

It is not intended that the foregoing treatment of depreciation in relation to public works and public finance should set out any new principles of depreciation or accountancy. Instead, a suggestion is offered that depreciation accountancy does have a place in government accounting procedures and finance. The full meal in any business of government is not paid for until capital costs are properly accounted for. This is particularly true of those functions or activities which produce a definite revenue, or which are in substitution for services or goods that may be, or logically could be, supplied by private sources. Until financial reports take full account of all items of annual costs, and account for investment in physical properties, they must be regarded as only partial truths.

In recent years there has been much talk of budgeting, long-range planning, and public spending to increase the level of business activity. Knowledge of the depreciation of public properties is essential to a wisely made plan involving any one of these three ideas. A government spending or tax budget, and a building program, could be arranged somewhat on an even keel over a period of years if attention were given to the rates of depreciation of the several properties of the public and the probable future needs. All too frequently, once money is spent for a public works or a public building, it is considered that no future obligation will exist. Government buildings, no less the same as for private buildings, decay, become functionally obsolete, and suffer for need of modernization. Depreciation studies will bring out these needs, determine rates of losses in values, and become the basis of estimating annual sums necessary to provide the citizens with physical properties necessary to the proper functioning of their government. Why, for instance, could not every state legislature, school board, and city council provide about a constant sum annually for building replacements and extensions? In the end the costs would be less and the burdens more evenly spread over generation to generation. Depreciation is inevitable; a good government must recognize and must budget for it in the same manner as for other annual expense.

Acknowledgment

The definitions of value and depreciation, under the heading "Value and Depreciation," were written by Anson Marston, Past-President and Hon. M. Am. Soc. C. E.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
PAPERS

STABILITY OF GRANULAR MATERIALS

By R. E. Glover,[1] Esq., and F. E. Cornwell,[2] Esq.

Synopsis

Results of studies made for the purpose of developing a theory of stability for granular materials, based directly on fundamental laws of behavior, are presented in this paper. Some elementary linear solutions are obtained which are applied to the problem of estimating the stability of earth dams.

Introduction

The primary purpose of soil mechanics is to develop methods of correlating laboratory tests with the performance of structures in the field. The correlation problems for granular materials are similar, in many respects, to those for elastic materials. Methods for calculating the performance of elastic structures have received much attention and the analytical tools for making the calculations have been grouped into a "theory of elasticity." Much of the theory of elasticity is built upon the postulates of equilibrium and of geometrical continuity subject to the relation between stress and strain imposed by Hooke's law. It is important to note that simplicity of the fundamental postulates does not insure a simple solution of the correlation problem since the elastic equations, in many cases, are difficult to solve.

The equations of equilibrium are as important for granular materials as for elastic materials but it can be seen readily that the continuity condition may not be appropriate for granular materials since these can often be sheared extensively without losing their properties.

In the elastic case, it has been proved[3] that a solution to the problem of stress distribution is unique provided it meets the foregoing equilibrium and continuity requirements and, in addition, satisfies the condition peculiar to the problem. These latter conditions are usually imposed at boundaries where conditions are known and, therefore, they are called "boundary conditions."

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **March, 1942.** Opening discussion on this paper appears elsewhere in this issue.

[1] Senior Engr., Bureau of Reclamation, Denver, Colo.

[2] Asst. Engr., Bureau of Reclamation, Denver, Colo.

[3] "A Treatise on the Mathematical Theory of Elasticity," by Augustus E. H. Love, University Press, Cambridge, England, 4th Ed., 1927, paragraph 118.

In the analysis to be described, equilibrium equations expressed in terms of stresses have been retained. The continuity condition is also retained where the stress system is such as will prevent relative movement between grains. Where slippage between grains is possible, the continuity condition has been replaced by a condition between the maximum and minimum principal stresses. This relation is the one commonly referred to as Coulomb's law, which states that resistance to shearing is compounded of an intrinsic cohesion, and a frictional resistance proportional to the normal stress acting on the plane of shear.

The trained engineer will find little that is new in this description and the methods to be described are not much more than a mathematical mobilization of relations with which the engineer has long been familiar. The inducement for the expenditure of effort, on analyses of the type described herein, comes from the possibility that the use of simplifying assumptions may be avoided and the reliability of the results thereby enhanced.

In developing a mathematical approach to the solution of any physical problem, one is faced with the necessity of reducing the problem to an exact status by means of specific definitions. It is usually necessary to replace the actual material, which is almost never continuous, by an ideal continuous material. Of course, it is desirable that, in so far as possible, the properties of the two materials be identical. In the present case, certain consequences of the granular structure of the actual material are imposed upon the ideal. Other limitations arise from purely mathematical difficulties.

In the theory of elasticity, stress is defined in terms of the resultant force transmitted across a small plane area within the body. In the present case it is expedient to modify this concept because of the presence within the voids of water under pressure. The term "stress," as used herein, is defined as the force transmitted by the contact pressures between grains over a small area divided by the area, and therefore excludes the percolation pressures.

To form the concept of stress in the granular material, pass a plane through it and find the surface that passes through no particle but which lies as close to the plane as possible. The particles on one side of the surface touch those on the other at several points, and at each of these points a force is transmitted from particle to particle. Now consider a small part of the area of the surface. The average stress on the small area is defined as the resultant of the forces transmitted from particle to particle across it, divided by the area of its projection on the plane. Conceptions of average normal and shear stresses can be formed in the same way. Since it is apparent that the diameter of the small area previously mentioned must be large compared to the diameter of a grain, in order to include enough contact points to yield a valid average, a stress computation made for a point, as designated by coordinates, will represent an average stress in the immediate vicinity of the point rather than a stress at a point.

The stress convention is shown in Fig. 1. Forces of gravity, flotation, and fluid friction, as well as an additional body force which is indicated as due to earthquake, are included. When it is desired to obtain the total force transmitted across a section, pore-pressure forces must be added.

The solution requires that every small element of volume must be in equilibrium under the stress resultants and forces that act upon it. In addition to equations of equilibrium, it is required that the stresses satisfy the equations of elasticity in those regions where the relation between stresses is such as will not permit grains to change their relative positions, and in the regions where this condition is not met, it is assumed that the relation between the principal stresses is such as to make slippage imminent on the planes of least resistance.

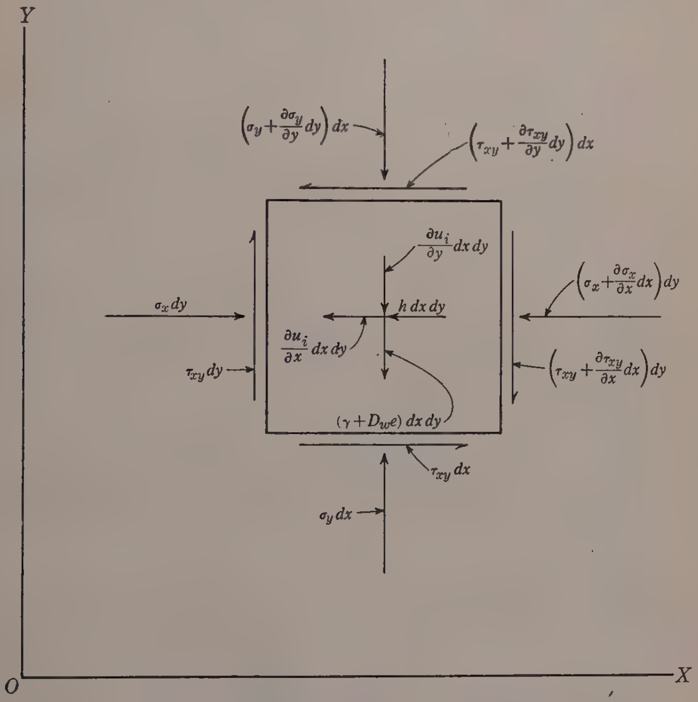

Y
$\left(\sigma_y+\frac{\partial\sigma_y}{\partial y}dy\right)dx$
$\left(\tau_{xy}+\frac{\partial\tau_{xy}}{\partial y}dy\right)dx$
$\frac{\partial u_i}{\partial y}dx\,dy$
$\left(\sigma_x+\frac{\partial\sigma_x}{\partial x}dx\right)dy$
$\sigma_x\,dy$
$h\,dx\,dy$
$\frac{\partial u_i}{\partial x}dx\,dy$
$\tau_{xy}\,dy$
$\left(\tau_{xy}+\frac{\partial\tau_{xy}}{\partial x}dx\right)dy$
$(\gamma+D_we)\,dx\,dy$
$\tau_{xy}\,dx$
$\sigma_y\,dx$
O
X

FIG. 1.—STRESS CONVENTION AND BODY FORCES

It should be noted that the terms "elastic" and "plastic" are used herein to describe zones in which previously stated relations between principal stresses exist and not as descriptions of types of material used in the structure. A solution consists of a distribution of stresses expressed as a function of the coordinates x and y which satisfies the conditions as stated.

NOTATION

The letter symbols used in this paper are defined where they first appear, and they are assembled for convenience of reference in Appendix I.

STRESS RELATIONS IN THE ELASTIC AND PLASTIC ZONES

Let σ be the normal stress on any plane, positive when compressive, and τ the shear stress. According to the theory of failure, which is accepted herein, it is true, for every plane at every point, that

$$|\tau| \leqq C + f\,\sigma \ldots\ldots\ldots\ldots (1a)$$

in which C and f are positive constants, known as the cohesion and the coefficient of friction, respectively, whose values depend upon the material; and it is also true that relative sliding of the material on the two sides of the plane is incipient when

$$|\tau| = C + f\,\sigma \ldots\ldots\ldots\ldots (1b)$$

but impossible when

$$|\tau| < C + f\,\sigma \ldots\ldots\ldots\ldots (1c)$$

Relations 1 may be written in terms of the function

$$R = C + f\,\sigma - |\tau| \ldots\ldots\ldots\ldots (2)$$

as follows: $R \geqq 0$, $R = 0$, and $R > 0$. In Eq. 2 R is a function of the position of the point and the orientation of the plane. Let R_0 be the minimum value of R at a given point. Then Eq. 1*a*, which is satisfied everywhere, is equivalent to $R_0 \geqq 0$.

Plastic Region.—At any point where $R_0 = 0$ there must be one or more planes; it is found that there are two, for which $R = R_0 = 0$, and along which, therefore, slipping is incipient. At such a point the material will be said to be in the "plastic state," and any region at every point of which the material is in the plastic state will be called a "plastic region."

Elastic Region.—At any point where $R_0 > 0$ there is no plane along which slipping is possible, since $R \geqq R_0 > 0$; and at such a point the material is assumed to obey Hooke's law and will be said to be in the "elastic state." Any region at every point of which the material is in the elastic state will be called an "elastic region."

As a consequence of Hooke's law, certain conditions of compatibility are satisfied at all points of an elastic region. For a two-dimensional body these conditions (which are discussed at length in works on the theory of elasticity) are reduced to the single equation:

$$\frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} - 2\,\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \ldots (3)$$

The Functions R and R₀.—Functions R and R_0 may be expressed in terms of the principal stresses or in terms of the normal and shear stresses. Let σ_1 and σ_2 be the principal stresses, positive when compressive, and σ_1 algebraically the greater; and let ϕ be the angle measured counterclockwise from the line of action of σ_1 to the plane of σ and τ. Equilibrium of the triangular element shown in Fig. 2 requires that

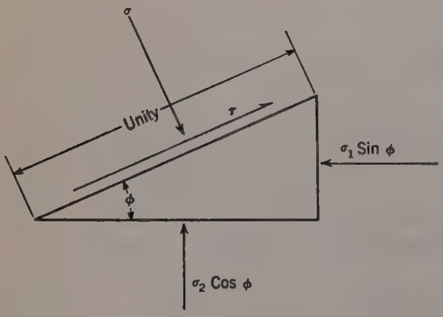


FIG. 2

$$\sigma = \sigma_1 \sin^2 \phi + \sigma_2 \cos^2 \phi \ldots\ldots\ldots\ldots (4a)$$

and

$$\tau = (\sigma_1 - \sigma_2) \sin \phi \cos \phi \tag{4b}$$

or

$$\sigma = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\,\phi \tag{5a}$$

and

$$\tau = \frac{\sigma_1 - \sigma_2}{2} \sin 2\,\phi \tag{5b}$$

After substituting Eqs. 5 in Eq. 2 and simplifying,

$$R = C + f\frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2}(f \cos 2\,\phi + |\sin 2\,\phi|) \tag{6a}$$

Then remembering that R_0 is the minimum of R at a given point, and that at a given point σ_1 and σ_2 are constants, one finds

$$R_0 = C + f\frac{\sigma_1 + \sigma_2}{2} - \sqrt{1 + f^2}\,\frac{\sigma_1 - \sigma_2}{2} \tag{6b}$$

This minimum is attained when

$$\phi = \pm \operatorname{arc\,tan}\,(\sqrt{1 + f^2} - f) \tag{7}$$

The two planes thus defined form the acute angle, arc tan $\frac{1}{f}$, which is bisected by the line of action of σ_1, the greater of the principal stresses. These are the planes along which slip occurs, if it does occur.

Other useful relationships, the derivations of which are omitted, are:

$$\frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_x + \sigma_y}{2} \tag{8a}$$

$$\frac{\sigma_1 - \sigma_2}{2} = \tfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \tag{8b}$$

$$\sigma_x = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\,\theta \tag{8c}$$

$$\sigma_y = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\,\theta \tag{8d}$$

$$\tau_{xy} = \frac{\sigma_1 - \sigma_2}{2} \sin 2\,\theta \tag{8e}$$

$$\tan 2\,\theta = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y} \tag{8f}$$

$$\sin 2\,\theta = \frac{2\,\tau_{xy}}{\sigma_1 - \sigma_2} \tag{8g}$$

and

$$\cos 2\,\theta = \frac{\sigma_x - \sigma_y}{\sigma_1 - \sigma_2} \tag{8h}$$

n which θ is the angle measured counterclockwise from the x-axis to the line of action of σ_1. It frequently happens that confusion arises in the computation of θ; and it should be remembered, therefore, that: $\sigma_1 - \sigma_2$ is never negative; θ is referred always to σ_1, the greater—never to σ_2, the lesser; and θ is defined not by the value of tan 2 θ (Eq. 8*f*), but by the value of tan 2 θ and the algebraic sign of either sin 2 θ or cos 2 θ. If Eqs. 8*a* and 8*b* are substituted in Eq. 6*b*, the following expression for R_0 is obtained:

$$R_0 = C + f\,\frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{1 + f^2}}{2}\,\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \quad \text{(9)}$$

COHESIONLESS MATERIAL

The types of solutions available at present are applicable only to a cohesionless material; that is, one for which $C = 0$. For such a material the law $R_0 \geqq 0$ becomes, in terms of the principal stresses, from Eq. 6*b*,

$$f\,\frac{\sigma_1 + \sigma_2}{2} \geqq \sqrt{1 + f^2}\,\frac{\sigma_1 - \sigma_2}{2} \quad \text{(10a)}$$

or, in terms of σ_x, σ_y, and τ_{xy}, from Eq. 9,

$$f\,\frac{\sigma_x + \sigma_y}{2} \geqq \frac{\sqrt{1 + f^2}}{2}\,\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \quad \text{(10b)}$$

By letting $B = \frac{\sqrt{1 + f^2}}{f}$ Eqs. 10 reduce to

$$\sigma_2 \geqq \frac{B - 1}{B + 1}\,\sigma_1 \quad \text{(11a)}$$

and

$$\sigma_x + \sigma_y \geqq B\,\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \quad \text{(11b)}$$

The equalities represented by Eqs. 11 are two ways of stating the plasticity condition, and the inequalities represent a relation that must be satisfied in elastic regions. These expressions imply, and it can be proved, that tensile stresses are impossible, that inside an elastic region neither of the direct stresses can vanish; also that inside or on the boundary of a plastic region, if either direct stress is zero, then all the stresses are zero. It follows that all stresses vanish at a free boundary, and it should be noted that any boundary subjected to water load alone is free, provided the water pressure is continuous across the boundary.

Boundary Between Elastic and Plastic Region.—It is clear that, across a boundary between an elastic and plastic region, the direct stress normal to the boundary and the shear stress tangential to the boundary are continuous, but it is not clear whether the direct stress tangential to the boundary is or is not continuous.

It appears that the theory must permit such a boundary to move as the loading on the outer boundaries of the body is varied. If the direct stress tangential to the boundary is not continuous, then, as the boundary moves, there must be sudden changes of stress at points on the boundary, and sudden movements of the particles, involving infinite accelerations. Such consequences are to be avoided, and it is postulated, therefore, that all stresses are continuous across the boundary. Of course, this postulate holds only in case the properties of the material are continuous across the boundary.

APPLICATION TO THE PROBLEM OF EARTH-DAM STABILITY

The structure to be analyzed is idealized by dividing it arbitrarily into a series of elastic and plastic regions (see Fig. 3). Odd Roman numerals (I, III, V, etc.) are used to designate regions assumed to be plastic and even ones (II, IV, VI, etc.) denote regions assumed elastic. In every case it is assumed that the inner boundaries, those joining two regions, are straight lines. The use, in the equations of inner boundaries, of arbitrary constants α and β with subscripts associating them with the elastic region involved is also illustrated. The corresponding terms for the outer boundaries (that is, numbers n_1 and n_2) used in the equations of the downstream and upstream slopes are constant, of course, for any one dam and are obtainable directly from known data.

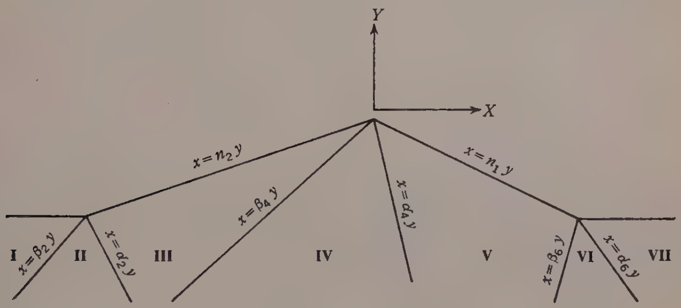


FIG. 3

Note that the problem, as illustrated, is cut into three parts in which are grouped regions I, II, III, regions III, IV, V, and regions V, VI, VII. The separate parts are referred to as applying, respectively, to the upstream toe, the dam, and the downstream toe. For each part a new system of axes, with its origin at the point common to all three regions and with the x and y axes parallel to and with the sense of those for regions III, IV, V, is used. Additional parts are needed when the slope of either fill is not uniform throughout. In such a case the actual slope is approximated by two or more sections of constant slope and then a new group is introduced at each point where there is a change.

EQUATIONS AND LINEAR SOLUTIONS FOR ELASTIC REGIONS

The forces acting on an element of the material are shown in Fig. 1. If the accelerations of the water are negligible, the equations of equilibrium are seen to be

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial u}{\partial x} + h = 0 \ldots\ldots\ldots\ldots\ldots\ldots (12a)$$

and

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial u}{\partial y} + \gamma + D_w\, e = 0 \ldots\ldots\ldots\ldots\ldots (12b)$$

The three partial differential equations (Eqs. 3 and 12) must be satisfied by any solution for an elastic region. Consider a definite region "i," and make

$$C_{2i} = \gamma + D_w\, e + \frac{\partial u_i}{\partial y} \qquad \text{(13a)}$$

and

$$C_{2i}\, C_{1i} = h + \frac{\partial u_i}{\partial x} \qquad \text{(13b)}$$

Then the general linear solution of Eqs. 3 and 12 is

$$\sigma_x = C_{2i}\,(a_{1i}\, x + a_{2i}\, y + a_{3i}) \qquad \text{(14a)}$$

$$\sigma_y = C_{2i}\,(b_{1i}\, x + b_{2i}\, y + b_{3i}) \qquad \text{(14b)}$$

and

$$\tau_{xy} = C_{2i}\,[-\,(1 + b_{2i})\, x - (a_{1i} + C_{1i})\, y + d_{3i}] \qquad \text{(14c)}$$

in which a_{1i}, a_{2i}, a_{3i}, b_{1i}, b_{2i}, b_{3i}, and d_{3i} are arbitrary constants.

In general, however, one does take the vertex or common point of all the regions involved as the origin of the coordinate system. With this in mind it is evident that all stresses must vanish at $x = 0$, $y = 0$. Then the solution in the elastic region "i" reduces to

$$\sigma_x = C_{2i}\,(a_{1i}\, x + a_{2i}\, y) \qquad \text{(15a)}$$

$$\sigma_y = C_{2i}\,(b_{1i}\, x + b_{2i}\, y) \qquad \text{(15b)}$$

and

$$\tau_{xy} = C_{2i}\,[-\,(1 + b_{2i})\, x - (a_{1i} + C_{1i})\, y] \qquad \text{(15c)}$$

In these expressions, the four arbitrary constants a_{1i}, a_{2i}, b_{1i}, and b_{2i} will be determined, along with α_i and β_i, by the boundary conditions along $x = \alpha_i\, y$ and $x = \beta_i\, y$.

The inequality $R_0 > 0$ expressed in Eq. 11b as

$$\sigma_x + \sigma_y > B\,\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \qquad \text{(16a)}$$

must be satisfied in order to comply with the definition of an elastic region.

EQUATIONS AND LINEAR SOLUTIONS FOR PLASTIC REGIONS

In any plastic region "i" a solution must satisfy two partial differential equations and one algebraic equation; namely, Eqs. 12 and the equation arising from the law $R_0 = 0$. This latter equation, called the plasticity equation, is from Eq. 11b,

$$\sigma_x + \sigma_y = B\,\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \qquad \text{(16b)}$$

The general linear solution of Eqs. 12 and 16b, in which C_{1i} and C_{2i} are defined by Eqs. 13, is

$$\sigma_x = -\,C_{2i}\,(C_{1i} + n - n^2\, b_{1i})\,(x - n\, y + \epsilon') \qquad \text{(17a)}$$

$$\sigma_y = C_{2i}\, b_{1i}\,(x - n\, y + \epsilon') \qquad \text{(17b)}$$

and

$$\tau_{xy} = -\,C_{2i}\,(1 - n\, b_{1i})\,(x - n\, y + \epsilon') \qquad \text{(17c)}$$

in which b_{1i} is determined by

$$(1 + n^2)^2 \, b^2_{1i} - 2\,[(n + C_{1i})\,(n^2 - 1 - 2\,f^2) + 4\,n\,(1 + f^2)]\,b_{1i} + (n + C_{1i})^2 + 4\,(1 + f^2) = 0 \tag{18}$$

except that the foregoing solution does not include the case corresponding to $n \rightarrow \infty$, which is

$$\sigma_x = C_{2i}\,a_{2i}\,(y + \epsilon') \tag{19a}$$

$$\sigma_y = - C_{2i}\,(y + \epsilon') \tag{19b}$$

and

$$\tau_{xy} = - C_{2i}\,C_{1i}\,(y + \epsilon') \tag{19c}$$

in which

$$a_{2i} = - \left[\frac{B^2 + 1}{B^2 - 1} \pm \frac{2\,B}{B^2 - 1} \sqrt{1 - C^2_{1i}\,(B^2 - 1)} \right] \tag{20}$$

Note that in each type of solution for a plastic region there are two cases, coming (for example) from the two solutions of the quadratic equation for b_{1i}. These are distinguished, whenever the two are different, by denoting that in which σ_x is the larger of the two values of σ_x as the passive case, the other as the active case.

As for elastic regions, the most important case is that in which the stresses are assumed to vanish at $x = 0$, $y = 0$. For such conditions,

$$\sigma_x = - C_{2i}\,(C_{1i} + n - n^2\,b_{1i})\,(x - n\,y) \tag{21a}$$

$$\sigma_y = C_{2i}\,b_{1i}\,(x - n\,y) \tag{21b}$$

and

$$\tau_{xy} = - C_{2i}\,(1 - n\,b_{1i})\,(x - n\,y) \tag{21c}$$

in which b_{1i} is determined by Eq. 18. Note that along $x = n\,y$ these expressions yield $\sigma_x = \sigma_y = \tau_{xy} = 0$. Another useful case where $C_{1i} = 0$ is that in which all stresses are zero along $y = 0$. Then for the active case,

$$\sigma_x = - C_{2i}\,\frac{B - 1}{B + 1}\,y \tag{22a}$$

$$\sigma_y = - C_{2i}\,y \tag{22b}$$

and

$$\tau_{xy} = 0 \tag{22c}$$

and for the passive case (in addition to Eqs. 22*b* and 22*c*)

$$\sigma_x = - C_{2i}\,\frac{B + 1}{B - 1}\,y \tag{23}$$

The condition that all three stresses vanish along the outer boundary is sufficient to exhaust the arbitrary constants in a plastic region. Hence the solution is then known completely in an "outer" plastic region—that is, in a plastic region part of whose boundary is part of the boundary of the structure.

Slip Lines.—The direction of the slip lines may be determined by computing the two angles θ and ϕ by formulas already presented. However, this computation may be shortened as follows: Let ψ equal the angle between the y-axis and a slip line, counted positive clockwise. Then the directions of the two families of such lines are,

$$\tan \psi_1 = \lambda_1 = - \frac{2 f \tau_{xy} + \sigma_y - \sigma_x}{2 \tau_{xy} - 2 f \sigma_y} \quad \text{(24a)}$$

and

$$\tan \psi_2 = \lambda_2 = - \frac{- 2 f \tau_{xy} + \sigma_y - \sigma_x}{2 \tau_{xy} + 2 f \sigma_y} \quad \text{(24b)}$$

Note that if the stresses are linear, as they are in the present solutions, then λ_1 and λ_2 are constants and the slip lines are two families of parallel straight lines. It is convenient to use the exact check formula,

$$\left| \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 \lambda_2} \right| = \frac{1}{f} \quad \text{(25)}$$

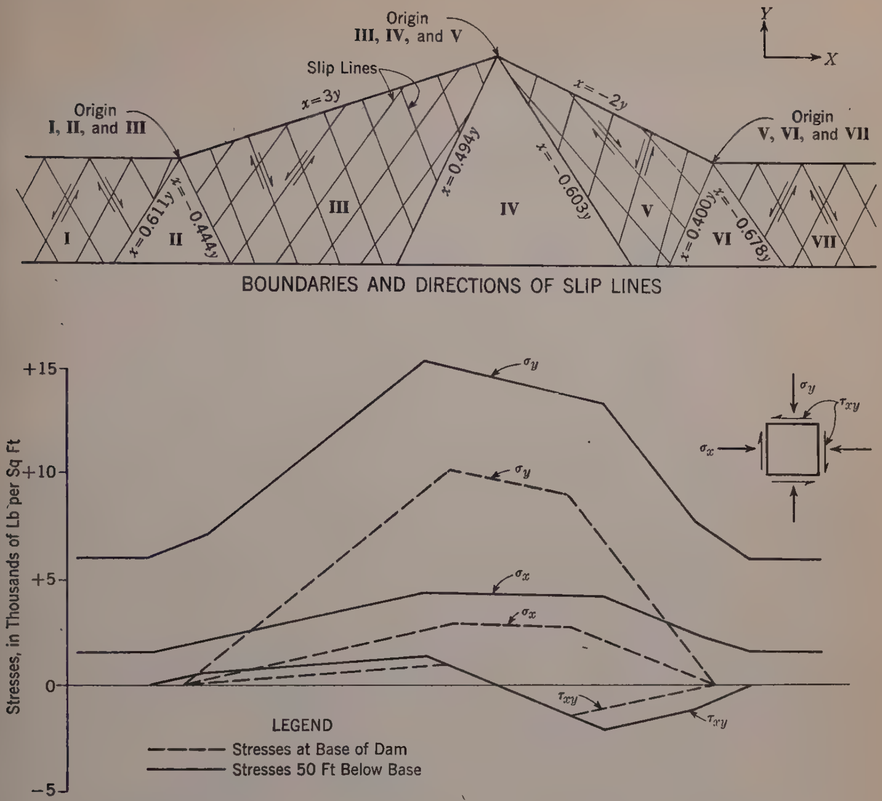
CONDITIONS FOR FAILURE

In the plastic zones the stress distribution is such that slippage is imminent. This does not mean, however, that a failure by sliding can occur since the direction of slip may be toward the foundation or an elastic zone where a reserve of strength is available. Under such condition, no slip of significant magnitude can occur and the structure is stable. The existence of a plastic zone then means simply that the material in the zone has acquired all the load it can carry. Any further load must be accommodated by encroachment on the reserve in the elastic zones by a readjustment of the boundary between the elastic and plastic zones. However, if a situation arises in which a continuous slip line can be drawn beginning in one free surface and terminating in another, then there is no limitation to movement and a slide may be expected. The condition for a sliding failure may then be reduced to a question of the possibility of drawing such a line. If a set of conditions is assumed which precludes the possibility of stability, no solution can be constructed. Such a condition can be readily obtained, for example, by choosing too steep a slope. In this case, the impossibility of constructing a solution is a definite indication of instability. Cases have occurred, however, in which a failure to obtain a solution may have indicated merely that no solution exists of the type sought. It is hoped that such questions may be answered as soon as more solutions to the fundamental equations are obtained.

The type of analysis described does not lead to an evaluation of a factor of safety but to a determination of stability or instability. An idea of the reserve strength available may be obtained, however, by investigating the reserve R in the elastic zones and by studying the solution to determine whether small or large changes of loading would be necessary to produce the conditions for a slide.

APPLICATIONS

Example 1. Solution of a Typical Uniform Material Dam of 1 on 3 Upstream and 1 on 2 Downstream Slopes, with Reservoir Empty.—Fig. 4 shows the analysis



BOUNDARIES AND DIRECTIONS OF SLIP LINES

DATA

Height of dam, in ft	100
Upstream slope	1 on 3
Downstream slope	1 on 2
Coefficient of friction, f	0.7
Dry density, in lb per cu ft, γ	120

EQUATIONS OF STRESSES

Regions I and VII:

$$\sigma_x = -0.271\,\gamma\,y \qquad \sigma_y = -\gamma\,y \qquad \tau_{xy} = 0$$

Region II:

$$\sigma_x = \gamma\,(0.066\,x - 0.311\,y) \qquad \sigma_y = \gamma\,(0.176\,x - 1.108\,y) \qquad \tau_{xy} = \gamma\,(0.108\,x - 0.066\,y)$$

Region III:

$$\sigma_x = 0.099\,\gamma\,(x - 3\,y) \qquad \sigma_y = 0.344\,\gamma\,(x - 3\,y) \qquad \tau_{xy} = 0.033\,\gamma\,(x - 3\,y)$$

Region IV:

$$\sigma_x = \gamma\,(-0.008\,x - 0.244\,y) \qquad \sigma_y = \gamma\,(-0.096\,x - 0.816\,y) \qquad \tau_{xy} = \gamma\,(-0.184\,x + 0.008\,y)$$

Region V:

$$\sigma_x = -0.171\,\gamma\,(x + 2\,y) \qquad \sigma_y = -0.543\,\gamma\,(x + 2\,y) \qquad \tau_{xy} = 0.085\,\gamma\,(x + 2\,y)$$

Region VI:

$$\sigma_x = \gamma\,(-0.129\,x - 0.359\,y) \qquad \sigma_y = \gamma\,(-0.281\,x - 1.190\,y) \qquad \tau_{xy} = \gamma\,(0.190\,x + 0.129\,y)$$

FIG. 4.—TYPICAL ANALYSIS FOR EMPTY RESERVOIR

as completed and also lists data used in the computations. An outline of the solution follows.

Upstream Toe.—For region I, $C_{11} = 0$ and $C_{21} = \gamma$. Stresses were computed by Eqs. 22.

For region III, $C_{13} = 0$, $C_{23} = \gamma$, $n = 3$. Stresses were computed by Eqs. 21, using the value of b_{13} from Eq. 18 for the active case.

For region II, $C_{12} = 0$, $C_{22} = \gamma$. Stresses for this region are given by Eqs. 15. In order to evaluate the four arbitrary constants in these equations and the two arbitrary constants in the equations of the boundaries, the direct and shear stresses were equated across the boundaries—that is, across $x = \alpha_2\, y$ and $x = \beta_2\, y$. This led to the six simultaneous equations,

$$\left.\begin{aligned} & a_{12}\,\beta_2 + a_{22} = -\,0.27108 \\ & b_{12}\,\beta_2 + b_{22} = -\,1 \\ & (1 + b_{22})\,\beta_2 + a_{12} = 0 \\ & a_{12}\,\alpha_2 + a_{22} = 0.09886\,(\alpha_2 - 3) \\ & b_{12}\,\alpha_2 + b_{22} = 0.34432\,(\alpha_2 - 3) \\ \text{and} \quad & \\ & (1 + b_{22})\,\alpha_2 + a_{12} = -\,0.03295\,(\alpha_2 - 3) \end{aligned}\right\}\cdots\cdots\cdots\cdots(26)$$

Eqs. 26 were solved and the numerical values placed in Eqs. 15. The resulting expressions for σ_x, σ_y, and τ_{xy} are given in Fig. 4. For a check of the inequality which must be satisfied in any elastic region (see Eq. 16*a*), the point $x = 0$, $y = -1$ was used, from which $\sigma_x = 0.311\,\gamma$, $\sigma_y = 1.108\,\gamma$, $\tau_{xy} = 0.066\,\gamma$, and $B = 1.7438$; then (see Eq. 16*a*) $1.409\,\gamma < \sigma_x + \sigma_y$ (as required).

Dam.—For region III the values determined in the computations for the upstream toe were applicable. This is true because of the fact that a shift in the origin of coordinates, to any point along the boundary where all stresses are zero, produces no change in the expressions for stresses. Stresses for region V were computed by the use of the same formulas and data as for region III except that $n = -2$. The stresses in region IV and expressions for the boundaries were established by equating direct and shear stresses across $x = \alpha_4\, y$ and $x = \beta_4\, y$. The inequality was checked as at the upstream toe.

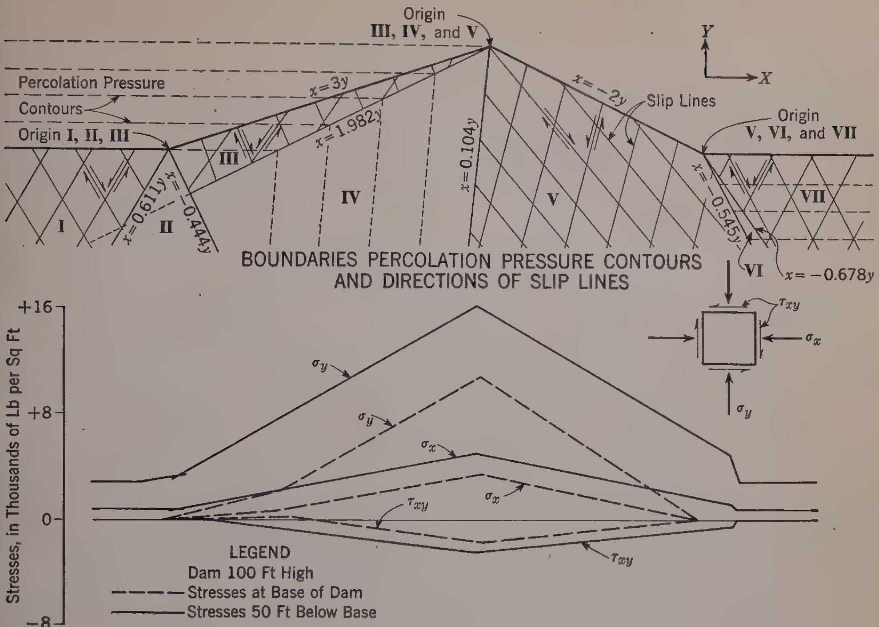
Downstream Toe.—Stress values for region V were obtainable directly from the solution for the dam. For region VII the stresses were the same as for region I at the upstream toe. The work of obtaining the stresses in region VI and of fixing the boundaries between regions was done by equating stresses as before, only this time the boundaries $x = \alpha_6\, y$ and $x = \beta_6\, y$ were used.

Directions of the slip lines as shown in the various plastic regions, Fig. 4, were obtained by substituting stresses for the region in question in Eqs. 24.

Example 2. Solution of a Typical Zoned Dam of 1 on 3 Upstream and 1 on 2 Downstream Slopes, with Reservoir Full.—Fig. 5 shows the results of the analysis and data used in the computations. An outline of the procedure follows.

Upstream Toe.—In this area the percolation pressure contours are taken as horizontal and as having a hydrostatic spacing vertically. The percolation function may then be written as $u = -\,D_w\,(y - H)$. This gives $\dfrac{\partial u}{\partial y} = -\,D_w$ and for a value of $\gamma + D_w\, e$ of 120 lb per cu ft; then, $C_2 = 57.6$; and $C_1 = 0$. The remainder of the solution is similar to that in Example 1.

Dam.—Stresses for region III may be taken from the upstream toe solution. Region V, which is taken as dry, was computed as in Example 1. In region IV



DATA

Height of dam, in ft	100
Dry density, in lb per cu ft	95
Wet density, in lb per cu ft	120
Coefficient of friction, f	0.7

EQUATIONS OF STRESSES

Regions I and VII:
$\sigma_x = -15.614\,y$
$\sigma_y = -57.6\,y$
$\tau_{xy} = 0$

Region II:
$\sigma_x = 3.788\,x - 17.930\,y$
$\sigma_y = 10.140\,x - 63.797\,y$
$\tau_{xy} = 6.197\,x - 3.788\,y$

Region III:
$\sigma_x = 5.695\,(x - 3\,y)$
$\sigma_y = 19.833\,(x - 3\,y)$
$\tau_{xy} = 1.898\,(x - 3\,y)$

Region IV:
$\sigma_x = 15.092\,x - 35.711\,y$
$\sigma_y = 46.984\,x - 113.32\,y$
$\tau_{xy} = -10.117\,x + 18.123\,y$

Region V:
$\sigma_x = -16.234\,(x + 2\,y)$
$\sigma_y = -51.558\,(x + 2\,y)$
$\tau_{xy} = 8.117\,(x + 2\,y)$

Region VI:
$\sigma_x = -60.195\,x - 56.423\,y$
$\sigma_y = -130.968\,x - 146.390\,y$
$\tau_{xy} = 88.790\,x + 60.195\,y$

FIG. 5.—TYPICAL ZONED DAM; ANALYSIS FOR FULL RESERVOIR

the percolation pressure contours were assumed parallel to the boundary $x = \alpha_4\,y$, so the percolation function was written as

$$u_4 = -D_w \frac{x - \alpha_4\,y}{\beta_4 - \alpha_4} \qquad (27)$$

Then

$$\frac{\partial u_4}{\partial y} = \frac{D_w\,\alpha_4}{\beta_4 - \alpha_4} \qquad (28a)$$

and

$$\frac{\partial u_4}{\partial x} = -\frac{D_w}{\beta_4 - \alpha_4} \qquad (28b)$$

When $\gamma + D_w\, e = 120$ lb per cu ft,

$$C_{24} = \frac{120\,\beta_4 - 57.6\,\alpha_4}{\beta_4 - \alpha_4} \qquad (29a)$$

and

$$C_{14} = \frac{-62.4}{120\,\beta_4 - 57.6\,\alpha_4} \qquad (29b)$$

The remainder of the solution is similar to that given in Example 1.

Downstream Toe.—Horizontal percolation pressure contours, spaced hydrostatic vertically, were taken in regions VI and VII. Then for these regions: $u = -D_w\, y$; $C_2 = 57.6$; and $C_1 = 0$.

Stresses in region V were computed for the dam assuming that it is dry. These same stresses were used when equating stresses across $x = \beta_6\, y$. This involves no inconsistency because the stresses computed for region VI are contact stresses and do not include the percolation pressure. The upward body forces due to flotation below river-bed level in zone V, however, are neglected. The details of the remainder of the procedure have been given previously.

By letting the percolation function in region VII be variable, its limiting value for stability may be determined by joining regions V and VII directly. This may be accomplished by equating normal and shear stresses across a common boundary and solving for the value of the pressure in the percolation function. This computation has been made for various conditions and the results are given in Table 3, Appendix II.

CONCLUSION

All present available solutions, of the fundamental equations that have been established in this paper, are linear in form. This definitely limits the stress systems in the various regions and makes mandatory a system of linear boundaries between regions. More general solutions are needed to avoid many of the present limitations. The present linear solutions should be considered as representing a first step in the development of a general method of approach.

ACKNOWLEDGMENTS

This method of analysis was developed in the Denver (Colo.) office of the Bureau of Reclamation, U. S. Department of the Interior, since about 1936. Among those who contributed were A. W. Adkins, E. D. Rainville, W. D. Dickenson, Jr., and W. H. Jurney. Technical studies of the office are conducted under the direction of Ivan E. Houk, M. Am. Soc. C. E., senior engineer. All designs and investigations are conducted under the supervision of J. L. Savage, M. Am. Soc. C. E., chief designing engineer. All engineering and construction work is under the general direction of S. O. Harper, M. Am. Soc. C. E., chief engineer, with headquarters in Denver, and all activities of the Bureau are under the direction of John C. Page, M. Am. Soc. C. E., commissioner, with headquarters at Washington, D. C.

APPENDIX I

NOTATION

The following symbols, used in this paper, conform essentially with "Soil Mechanics Nomenclature" approved by the Society in 1941 and published as Manual of Engineering Practice No. 22:

a = arbitrary constant (a_{1i}, a_{2i}, a_{3i}, etc.);

$B = \frac{\sqrt{1 + f^2}}{f}$;

b = arbitrary constant (b_{1i}, b_{2i}, b_{3i}, etc.);

C = cohesion; also arbitrary constant (C_{1i}, C_{2i}, C_{3i}, etc.);

D_w = density of water;

d = arbitrary constant (d_{1i}, d_{2i}, d_{3i}, etc.);

e = ratio of voids to total volume;

f = coefficient of internal friction;

H = depth of water in reservoir;

h = horizontal earthquake acceleration force expressed as a force per unit of volume;

n_1, n_2, n_3, etc. = arbitrary constants in the equations of the outer boundaries;

R = a function representing the reserve strength to resist shear failure;

R_0 = the minimum of R;

u = percolation pressure;

x and y = rectangular coordinates;

α = arbitrary constant in the equations of the inner boundaries;

β = arbitrary constant (see α);

γ = dry weight per unit of volume;

ϵ' = an arbitrary constant;

θ = angle of the x-axis to the line of action of σ_1, counted positive counterclockwise;

λ = tangent of the angle ψ;

σ = normal component of stress, positive when compressive;

σ_x and σ_y = normal stresses on planes normal to the x and y axes respectively;

τ = shear stress;

τ_{xy} = shear stress in direction of x on plane normal to y;

ϕ = angle from the line of action of σ_1 to the plane under consideration, counted positive counterclockwise;

ψ = angle from the y-axis to the slip lines, counted positive clockwise.

APPENDIX II

TABLE 1.—LIMITING SLOPES FOR VARIOUS EARTHQUAKE INTENSITIES

Earthquake acceleration—fraction of gravity	Wet density, lb per cu ft	COEFFICIENT OF FRICTION						
		0.50	0.55	0.60	0.65	0.70	0.75	0.80
0.00 *g*	115	4.63	4.25	3.94	3.69	3.47	3.28	3.11
	120	4.41	4.06	3.76	3.51	3.30	3.12	2.97
	127.5	4.15	3.81	3.54	3.30	3.11	2.94	2.79
	135	3.94	3.62	3.36	3.13	2.95	2.79	2.65
	142.5	3.77	3.46	3.21	3.00	2.82	2.66	2.53
0.05 *g*	115	5.99	5.37	4.89	4.49	4.17	3.90	3.67
	120	5.64	5.07	4.61	4.24	3.94	3.69	3.47
	127.5	5.22	4.70	4.29	3.95	3.67	3.44	3.24
	135	4.90	4.41	4.03	3.72	3.46	3.24	3.05
	142.5	4.64	4.19	3.82	3.53	3.29	3.08	2.90
0.10 *g*	115	8.41	7.23	6.37	5.71	5.19	4.77	4.43
	120	7.74	6.69	5.92	5.32	4.85	4.47	4.16
	127.5	6.99	6.08	5.40	4.87	4.46	4.11	3.83
	135	6.43	5.62	5.01	4.53	4.15	3.84	3.58
	142.5	6.00	5.26	4.70	4.26	3.91	3.62	3.38
0.15 *g*	115	13.91	10.93	9.05	7.76	6.82	6.11	5.55
	120	12.18	9.75	8.18	7.07	6.25	5.63	5.13
	127.5	10.43	8.52	7.23	6.31	5.62	5.09	4.66
	135	9.25	7.65	6.56	5.76	5.16	4.68	4.30
	142.5	8.40	7.02	6.05	5.34	4.80	4.37	4.03
0.20 *g*	115	38.58	21.79	15.33	11.92	9.81	8.38	7.35
	120	27.73	17.60	13.00	10.38	8.69	7.51	6.64
	127.5	20.11	13.97	10.79	8.84	7.53	6.59	5.88
	135	16.18	11.81	9.37	7.81	6.73	5.94	5.34
	142.5	13.77	10.38	8.39	7.08	6.15	5.46	4.93
0.25 *g*	115	....	....	47.35	24.87	17.06	13.10	10.70
	120	....	81.50	30.53	19.04	13.97	11.11	9.29
	127.5	....	37.14	20.66	14.47	11.22	9.23	7.89
	135	59.76	25.07	16.06	11.93	9.56	8.03	6.95
	142.5	36.42	19.44	13.40	10.32	8.44	7.19	6.29

Table 1 can be used to investigate the stability of slopes in the following cases:

(*a*) Downstream slopes of dams below the point where the phreatic line emerges;

(*b*) Upstream slopes following a rapid draw-down; and

(*c*) Slopes of cuts or fills when saturated.

TABLE 2.—LIMITING SLOPES FOR VARIOUS VALUES OF C_1

f	VALUES OF C_1:										
	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	2.00	2.28	2.63	3.07	3.67	4.50	5.75	7.83	12.00	24.5	....
0.55	1.82	2.06	2.34	2.71	3.17	3.79	4.66	5.96	8.13	12.48	25.50
0.60	1.67	1.87	2.12	2.42	2.80	3.29	3.93	4.84	6.20	8.47	13.00
0.65	1.54	1.72	1.94	2.20	2.51	2.91	3.41	4.09	5.04	6.46	8.83
0.70	1.43	1.59	1.78	2.01	2.28	2.61	3.03	3.56	4.27	5.26	6.75
0.75	1.33	1.48	1.65	1.85	2.09	2.38	2.72	3.16	3.71	4.46	5.50
0.80	1.25	1.39	1.54	1.72	1.93	2.18	2.48	2.84	3.30	3.89	4.67

The type of failure contemplated is shown in Fig. 6(*a*) to consist of a sliding of material down the slope. The fill material is assumed to be cohesionless. The percolation pressures are assumed to have a hydrostatic distribution in the vertical direction. The numbers in Table 1 represent the steepest slopes that can be expected to remain stable under assumed conditions.

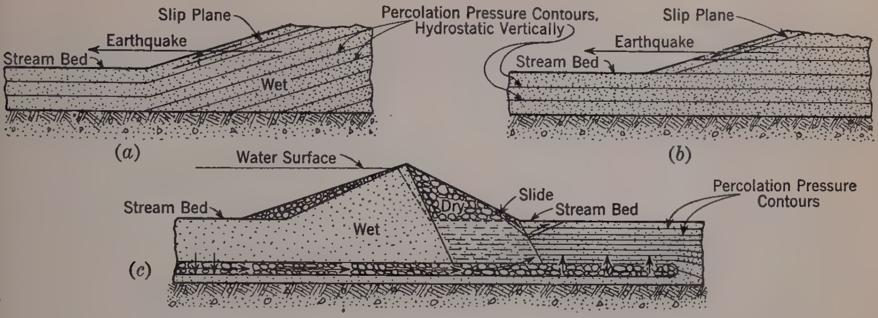


Fig. 6

Table 2 is useful for investigating either the stability of upstream slopes of earth dams under "reservoir full" conditions or the stability of dry slopes.

TABLE 3

Density, in lb per cu ft		Coefficient of Friction				Coefficient of Friction			
dry	wet	0.50	0.60	0.70	0.80	0.50	0.60	0.70	0.80
(*a*) Slope on Downstream Face, 1 on 2						(*c*) Slope on Downstream Face, 1 on 4			
90	115	97.78	104.89	108.02	110.03	101.55	105.61	108.32	110.16
100	120	100.87	108.76	112.25	114.47	105.02	109.56	112.57	114.62
110	127.5	106.45	115.14	118.97	121.42	111.06	116.02	119.33	121.59
120	135	112.04	121.51	125.70	128.37	117.07	122.47	126.09	128.55
130	142.5	117.63	127.89	132.42	135.32	123.07	128.93	132.85	135.51
(*b*) Slope on Downstream Face, 1 on 3						(*d*) Horizontal			
90	115	101.22	105.47	108.25	110.13	115.00	115.00	115.00	115.00
100	120	104.69	109.41	112.51	114.59	120.00	120.00	120.00	120.00
110	127.5	110.66	115.85	119.26	121.55	127.50	127.50	127.50	127.50
120	135	116.63	122.30	126.01	128.51	135.00	135.00	135.00	135.00
130	142.5	122.60	128.74	132.76	135.47	142.50	142.50	142.50	142.50

The type of failure contemplated is shown in Fig. 6(*b*) to consist of a sliding of material down the slope. The fill material is assumed cohesionless. The percolation pressures for saturated slopes are assumed to have a hydrostatic distribution in the vertical direction. The numbers in Table 2 represent the steepest slopes that can be expected to remain stable under assumed conditions.

The quantity C_1 is defined for wet material as

$$\frac{\text{Fraction of gravity} \times \text{wet density}}{\text{Wet density} - \text{density of water}},$$

and for dry material as fraction of gravity.

Table 3 is useful for evaluating the effects of excessive uplift or percolation pressure gradients at the downstream toe of earth dams. It is also useful for estimating the intensity of the upward percolation pressure gradients necessary to produce piping.

The computations indicate that a slide should occur at the downstream toe before actual piping is observed. However only a small difference is indicated between the gradients necessary to cause sliding and piping, and, if the conditions were local, it would be reasonable to expect the piping phenomena to appear first.

The individual entries in Table 3 are the upward percolation pressure gradients in pounds per cubic foot necessary to produce failure, as shown in Fig. 6(*c*). The gradients necessary to produce piping are indicated in that part of the table marked "Horizontal."

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
PAPERS

TIMBER FRICTION PILE FOUNDATIONS

By Frank M. Masters,[1] M. Am. Soc. C. E.

Synopsis

The theory of timber friction pile behavior presented in this paper was developed in connection with pile tests on the Morganza Floodway in Louisiana. It may afford a means of determining, with reasonable accuracy, the safe loads to impose on groups of piles in any arrangement, and of any length, from data secured by the loading to failure of one or more individual friction piles at the site.

Introduction

Many large areas of land in the United States on which structures imposing heavy foundation loads are constructed are formed of silty and clayey materials, being delta deposits of the larger streams. Sandy deposits or other such firm materials suitable for the support of heavy concentrated foundation loads are frequently found at such great depths below the surface of these deposits that it is not economically possible to construct foundations that bear directly on the firm materials, or that are supported on point bearing piles driven into these firm materials. Moreover, timber piles, which are relatively inexpensive and of remarkable durability when embedded entirely below the ground water, are becoming increasingly scarce in the longer lengths required to reach such firm bearing strata. Consequently, many existing structures are supported on piles embedded in the overlying silt and clay materials. It is becoming more and more important, therefore, to find some reasonably accurate method of determining the safe carrying capacity of single piles and groups of piles that do not reach point bearing in a firm stratum, but depend instead for their support entirely upon the adhesion of the surrounding clay or silt to the embedded pile surfaces.

Such piles, popularly called "friction" piles, are the subject of analysis in this paper. The adhesion that supports them transmits their load to the soil by vertical shearing stresses, which are greatest in the soil at the periphery of the single pile or of the pile group.

Note.—Written comments are invited for immediate publication ; to insure publication the last discussion should be submitted by March, 1942.

[1] Cons. Engr. (Modjeski & Masters), Harrisburg, Pa.

The tip of a "friction" pile delivers practically no load in bearing, so that the load-carrying capacity of a single pile is dependent upon the length of the pile, the diameter and taper of the pile, and the shearing strength of the soil. The last may increase somewhat from the surface downward due to the additional internal friction generated in the soil on account of horizontal earth pressures from the weight of the overlying materials. However, due to the fact that the butt diameter of timber piles is larger than the tip diameter, the load delivered to the soil per unit of length of a timber pile will usually be greatest at the butt and smallest at the tip.

Load tests have shown that pile-driving formulas, in which the weight of the ram and the blows per foot of penetration are used to determine the carrying capacity of the pile, do not apply to friction piles, probably because the characteristics of clay or silt during driving are different from what they are when the pile is at rest in the clay or silt. Static load tests, which simply measure the load that creates a predetermined amount of settlement within a fixed amount of time, also do not properly measure the ultimate carrying capacity of friction piles, which is measured rather by the shearing value of the soils in which the piles are embedded than by the compressibility of the underlying materials.

It has long been known that an average individual pile in a group arrangement of friction piles does not have the same carrying capacity as an isolated single pile and that the arrangement and spacing, center to center, of the piles in a group affect the load capacity of the piles. Because of the lack of tests to failure of group arrangements of friction piles, compared with individual friction piles driven in the same character of soil, there has been little or no comparable information available concerning the supporting capacity of single piles and piles in group arrangements with varying spacing, from which a theory might be developed for the economic design of friction pile footings.

THE MORGANZA PILE TESTS

As a part of the Mississippi River flood-control program, the U. S. Engineer Corps is required to construct railway and highway structures across the Morganza Floodway in Louisiana. In order to determine the most economical design of these structures, which are being erected by the railway lines, the Engineer Corps arranged with the railway companies to make tests of various types of piles on which to support the proposed trestles. The results of these tests have been made available by the Corps of Engineers for use in this paper. The locations of these tests, along the lines of the proposed railway structures, with relation to borings and subsurface investigations, are shown in Fig. 1. It excludes those tests carried to point bearing and shows only the tests of timber and concrete friction piles that were loaded to failure. The soil above the "sand line" in Fig. 1 is of approximately uniform quality along the Floodway, consisting of successive strata of loam, silt and clay, and mixtures of those materials. At the Port Allen (La.) Branch Crossing (not shown in Fig. 1), pile C22, Station 2348 + 70, has 77.5 ft of penetration, with its tip just reaching the sand line. The driving records show that until a pile had reached the sand stratum, where it came up suddenly, the blows per foot increased only gradually with penetration, and the borings and water-content records all indicate a

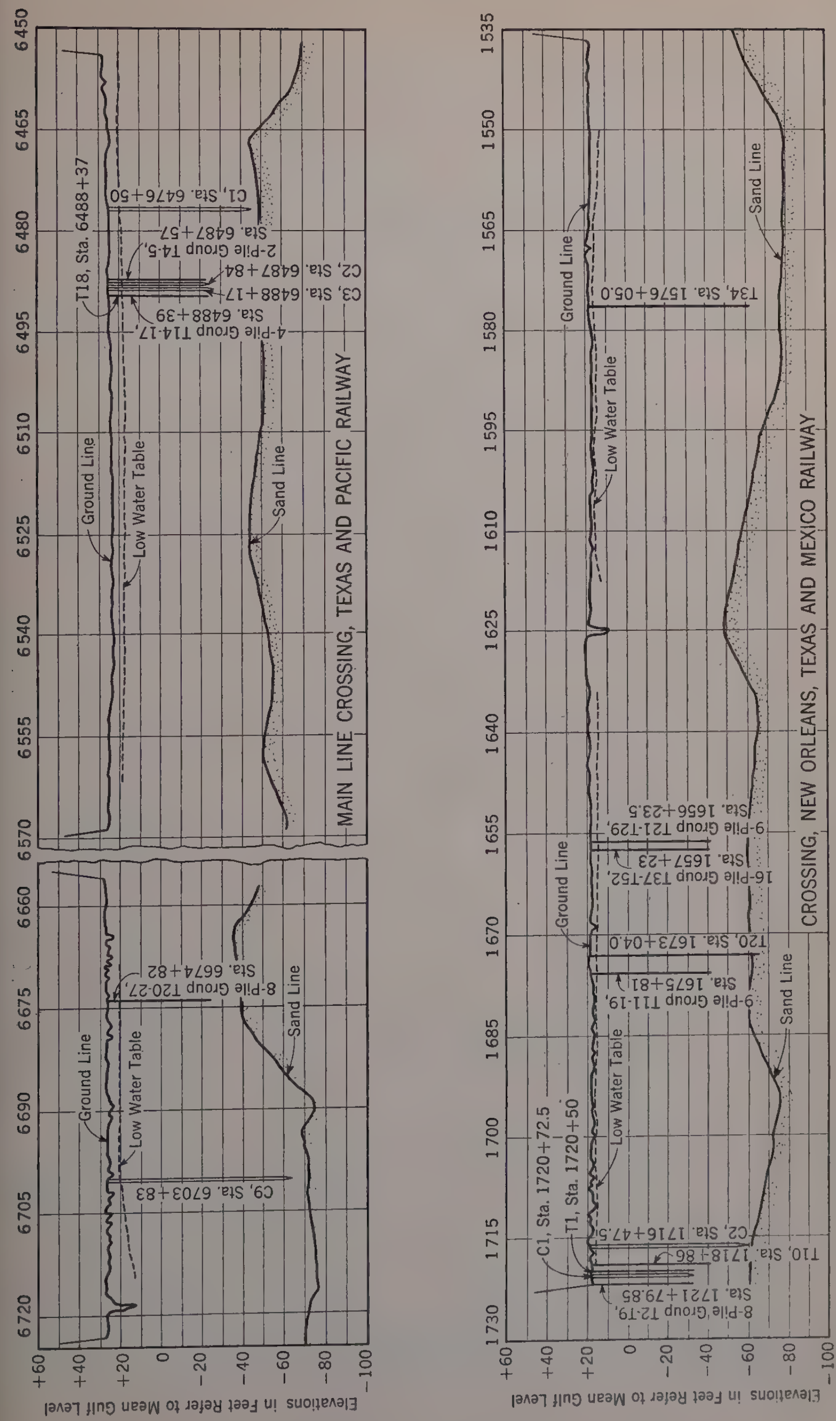


FIG. 1.—PROFILES OF RAILWAY CROSSINGS OF THE MORGANZA FLOODWAY SHOWING LOCATIONS OF FRICTION PILES TESTED TO FAILURE

fairly homogeneous material into which the test piles were driven at the various locations. All of the materials above the sand stratum are of silty, clay-like composition.

The pile sizes, driving records, and failure loads of the friction piles tested to failure, both timber and concrete, are given in Table 1. Actually, many tests, not recorded herein, were carried to a load just sufficient to meet the requirements of standard specifications for timber pile tests—that is, to a load that would not produce a settlement of more than ¼ in. in a period of forty-eight hours. However, some of the early group arrangements when so loaded settled beyond these specification requirements, and in so failing clearly indicated that piles in a group did not have the same supporting capacity as individual piles. Accordingly, the Corps of Engineers, when requested, agreed to load to failure one additional individual pile, and two 9-pile groups, all of which had been previously partly loaded, and in addition to drive and load a new

TABLE 1.—SUMMARY OF PILE SIZES, AND DRIVING AND LOADING DATA, FOR FRICTION PILES TESTED TO FAILURE

Pile[a]	Type[b]	DIAMETER, IN IN.[c]		Weight[d]	Hammer No.[e]	BLOWS PER FOOT		Formula safe load (tons)[g]	Penetration (ft)	Average contact area (sq ft)	Average settlement (ft)[h]	AT FAILURE	
		Butt	Tip			Half length[f]	Last foot					Load[i]	Skin friction[j]
(*a*) TEXAS AND PACIFIC RAILWAY; MAIN LINE CROSSING (T. & P.)													
C1	27-in. HOC	...	...	28.8	OR	10	83	48	52.2	389	2.021	117.5	604
C2	24-in. SOC	...	...	29.8	OR	11	22	46	50.5	335	1.826	117.5 –	702
C3	30-in. HOC	...	...	35.6	OR	21	29	59	50.5	418	2.513	117.5	562
C9	30-in. HOC	...	...	63.2	OR	34	314	219	92.0	762	0.032	210	551
T4–5	UYP	16.56	8.87	2.25	2	16	27	13	50.0	166	0.009	60	721
T14–17	YP and cypress	16.80	8.35	2.1	2	15	27	14	50.0	165	Tipped	50	610
T18	YP	16.50	8.50	2.1	2	13	25	12	50.5	165	1.510	70	847
T20–27	YP	15.91	8.10	2.1	2	9	20	10	50.0	158	0.049	40	510
(*b*) PORT ALLEN BRANCH CROSSING													
C22	30-in. HOC	...	...	52.7	OR	28	452	238	77.5	609	0.351	210	688
(*c*) NEW ORLEANS, TEXAS AND MEXICO RAILWAY CROSSING (N.O.T. & M.)													
C1	30-in. HOC	...	...	35.4	0	22	58	79	51	415	0.360	87.5	420
C2	30-in. HOC	...	...	62.6	0	42	493	196	81.5	669	1.329	127.5	380
T1	UYP	17.50	8.50	2.2	1	4	16	18	52.0	195	Failed	66.5	685
T2–9	UYP	16.91	10.20	2.3	1	6	11	13	50.4	179	0.042	42.5	476
T10	UYP	17.75	11.50	3.8	1	6	13	15	59.5	214	Failed	96.5	900
T11–19	UYP	18.08	10.67	3.2	1	6	14	16	59.7	219	0.231	56.3	520
T20	UYP	17.50	8.00	3.2	1	3	30	30	85.0	295	0.023	100	680
T21–29	UYP	18.08	10.50	3.0	1	5	11	13	59.9	224	0.032	55.33	495
T34	CYP	16.75	8.50	3.1	1	3	8	9	83.0	283	Failed	110	777
T37–52	UYP	16.86	8.31	2.35	1	4	11	13	60.0	203	0.236	42.9	423

[a] C = "concrete"; T = "timber"; T37–52, etc., = timber piles T37 to T52; that is, a 16-pile group. [b] YP = yellow pine; U = untreated; CYP = creosoted yellow pine; HOC = hollow octagonal concrete pile; and SOC = solid octagonal concrete pile. [c] Average for the group. [d] In thousands of pounds per pile. [e] Vulcan type of hammer; OR = manufacturer's number. [f] Blows per foot with pile half driven. [g] Safe load, in tons per pile, by "Engineering News" formula. [h] Average permanent settlement, after unloading, in feet. [i] In tons per pile. [j] In pounds per square foot of contact area.

16-pile group so as to obtain information that would show the carrying capacity of single piles of different lengths as well as groups of piles in different arrangements and spacings. The behavior of these three groups was similar; each

TABLE 2.—TEST RECORDS FOR INDIVIDUAL TIMBER FRICTION PILES, LOADED TO FAILURE; N.O.T. & M. CROSSING[a]

(a) PILE T10; 59.5 FT LONG

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
0	0	0
16.5	1:00	0.002
	5:45	0.002
31.5	6:00	0.004
	10:00	0.005
	14:00	0.006
	21:00	0.006
41.5	21:30	0.007
	24:00	0.009
	29:00	0.009
51.5	32:00	0.013
	35:00	0.013
	38:00	0.014
	41:00	0.015
	56:30	0.015
	61:00	0.014
	69:00	0.014
	71:45	0.015
	75:20	0.015
46.5	75:20	0.013
41.5	75:20	0.012
36.5	75:20	0.012
31.5	75:20	0.011
26.5	75:20	0.010
21.5	75:20	0.010
16.5	75:30	0.009
21.5	75:50	0.010
26.5	75:50	0.011
31.5	75:50	0.011
36.5	75:50	0.011
41.5	75:50	0.012
46.5	75:50	0.012
51.5	75:50	0.013
56.5	76:00	0.013
	79:00	0.013
	82:30	0.015
	91:00	0.015
	94:45	0.013
61.5	94:55	0.013
	97:00	0.016
	99:30	0.020
66.5	100:15	0.020
	109:00	0.020
	113:00	0.021
	119:45	0.021
71.5	120:00	0.022
	122:00	0.022
	124:00	0.023
	125:50	0.023
76.5	126:00	0.023
	130:00	0.024
	134:00	0.025
	142:00	0.025

(b) PILE T18; 50.5 FT LONG

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
0	0	0
10	0	0.004
	4:00	0.004
15	4:00	0.006
	8:00	0.006
20	8:00	0.008
	10:00	0.008
	12:00	0.009
25	12:00	0.009
	14:00	0.009
	16:00	0.010
30	16:00	0.011
	18:00	0.012
	20:00	0.012
35	20:00	0.013
	22:00	0.013
	24:00	0.015
40	24:00	0.017
	26:00	0.017
	28:00	0.018
45	28:00	0.018
	32:00	0.018
50	32:00	0.021
	36:00	0.021
55	36:00	0.026
	38:00	0.026
	40:00	0.027
60	40:00	0.028
	42:00	0.029
	44:00	0.031
65	44:00	0.032
	46:00	0.037
	48:00	0.038
70[b]	48:00	0.039
	49:00	1.510

(a) PILE T10 (continued)

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
81.5	142:50	0.027
	144:00	0.028
86.5	144:30	0.028
	147:00	0.029
	149:00	0.030
91.5	149.00	0.031
96.5	149:10	0.031
	149:30	0.034
	149:53	0.040
	149:55[c]	....
....	....	....
....	....	....

(c) PILE T20; 85 FT LONG[d]

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
0	0	0
7.5	0	0
15.0	0:25	0.003
25.0	0:35	0.003
	4:00	0.004
	8:00	0.004
	12:00	0.005
	16:00	0.005
	18:25	0.006
30.0	18:30	0.008
40.0	18:35	0.009
50.0	18:40	0.013
	22:00	0.015
	32:01	0.015
	36:00	0.016
	41:50	0.016
60.0	42:00	0.016
70.0	42:10	0.019
75.0	42:15	0.020
	44:01	0.024
	48:30	0.025
	52:00	0.025
	56:01	0.026
	65:25	0.026
80.0	65:35	0.027
90.0	65:45	0.028
100.0	66:00	0.028
	69:30	0.032
	73:00	0.033
	77:00	0.034
	83:00	0.034
	86:00	0.035
	104:01	0.035
	108:00	0.036
	116:01	0.036
	142:00	0.042
	161:00	0.044
	185:00	0.045
80.0	185:25	0.043
70.0	185:28	0.042
60.0	185:30	0.040
50.0	185:34	0.038
40.0	185:40	0.035
30.0	188:10	0.029
25.0	188:13	0.028
15.0	188:18	0.027
7.5	188:20	0.027
0	188:25	0.026
....	198:00	0.023
....	....	....
....	....	....
....	....	....
....	....	....

(d) PILE T1; 52 FT LONG

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
0	0	0
11	0	0.003
	4:00	0.003
16.5	4:00	0.005
	6:00	0.006
	8:00	0.006
21.5	8:00	0.008
	10:00	0.010
	12:00	0.010
26.5	12:00	0.011
	16:00	0.011
31.5	16:00	0.012
	18:00	0.013
	20:00	0.014
36.5	20:00	0.017
	22:00	0.018
	24:00	0.018
41.5	24:00	0.020
	28:00	0.020
46.5	28:00	0.024
	30:00	0.025
	32:00	0.025
51.5	32:00	0.038[e]
	34:00	0.039[e]
	36:00	0.041[e]
56.5	36:00	0.038[e]
	38:00	0.040[e]
	40:00	0.040[e]
61.5	40:00	0.029
	42:00	0.030
	44:00	0.030
66.5	44:00	0.051
	46:00	0.058
	48:00	....[c]

(e) RETEST, PILE T34; 83 FT LONG

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
0	0	0
30	0	....
55	....	....
80	21:45	0.047
	24:45	0.051
	29:25	0.051
90	29:25	0.056
	38:25	0.059
100	43:55	0.063
	53:35	0.063
110	53:35	....[f]
	57:25	0.199
	70:25	0.239[g]

[a] Except pile T18 (Table 2(*b*)), which is T. & P. Crossing. Several intermediate readings where there were no time-settlement changes have been omitted for brevity. [b] This pile settled under 70 tons until the bracing system rested on the ground. [c] Failure. [d] Pile T20 did not plunge sharply; but because it continued to settle under the 100-ton load, at least partial failure occurred. [e] Readings incorrect; pile not plumb. [f] Rapid settlement. [g] Further settlement was prevented by the shoring under the loading blocks.

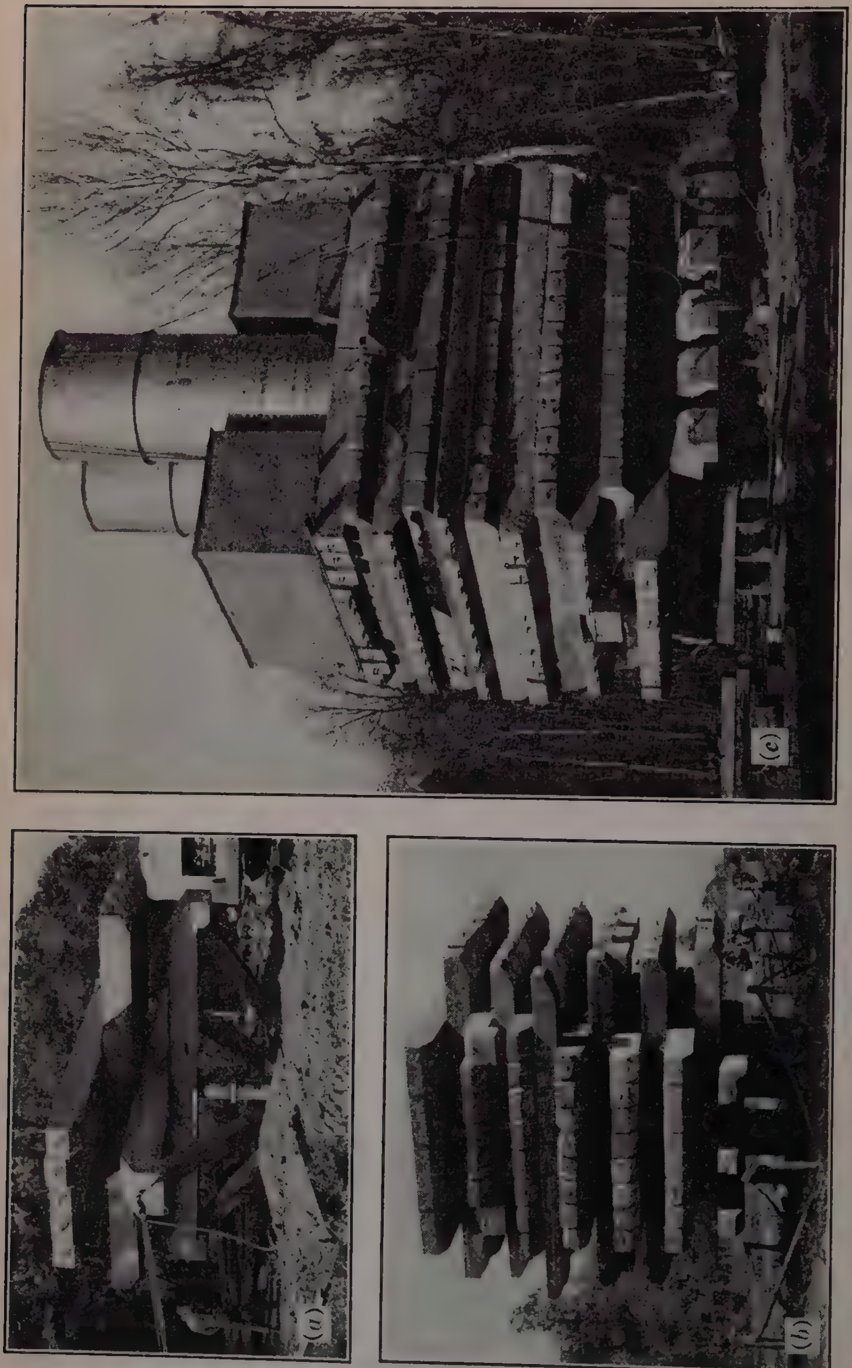


Fig. 2.—Pile Tests to Failure: (*a*) Pile T1, N.O.T. & M., After Failing Under a 66.5-Ton Load; (*b*) 9-Pile Group at Station 1656 + 23.5 (see Fig. 1), Under a Total Load of 498 Tons; (*c*) 16-Pile Group at Station 1657 + 23 (Fig. 1), Under a Total Load of 775 Tons

failed by leaning toward a corner or a side generally having the smallest diameter piles, and in failing carried down the earth surface with it in a dished depression extending outside the group and having a maximum value near the group center.

TABLE 3.—TEST RECORDS FOR GROUPS OF TIMBER FRICTION PILES LOADED TO FAILURE

N.O.T. & M. CROSSING[a]

(a) PILES T21–T29; NINE, 60-FT PILES

Total load per pile (tons) (1)	Time: Days (2a)	Time: Hr:min (2b)	Settlement, in ft (3)
0	0	0	0
5	0	2:30	0.002
10	0	3:15	0.003
	0	8:00	0.004
	0	12:00	0.004
	0	16:00	0.005
	1	1:30	0.005
15	1	1:30	0.007
	1	5:00	0.008
	1	19:00	0.009
	1	23:00	0.009
20.3[b]	1	23:30	0.011
20.3[b]	243	0	0.011
25.3	243	0	0.012
30.3	243	5:00	0.012
	243	22:00	0.016
	244	4:00	0.016
	245	4:45	0.0162
35.3	245	4:45	0.0166
	245	19:00	0.0176
	246	2:30	0.0186
40.3	247	3:00	0.0205
	247	22:00	0.0219
	248	5:00	0.0219
45.3	248	5:00	0.023
	248	21:00	0.0242
	249	22:00	0.0266
	250	23:00	0.0286
50.3	251	5:00	0.0303
	251	19:00	0.0329
	252	0	0.0349
	252	23:00	0.0361
	254	3:00	0.0376
	254	23:00	0.0376
55.3	255	0:30	0.0416
	255	3:30	0.045
	255	22:00	0.0565[c]

(b) PILES T2–T9; EIGHT, 50.5-FT PILES

Total load per pile (tons) (1)	Time: Days (2a)	Time: Hr:min (2b)	Time, in hr (2)	Settlement, in ft (3)
0	0	0		0
10.3	0	3:00		0.006
10.3			7	0.008
			10	0.009
			14	0.0095
			17	0.010
15.3			18	0.012
			21	0.012
			24	0.014
			26	0.014
			29	0.015
20.3			30	0.017
			34	0.018
20.3			38	0.0185
			41	0.019
25.3			42	0.023
			46	0.025
			53	0.025
30.3			55	0.028
			58	0.030
			61	0.031
			64	0.032
35.3			66	0.035
			69	0.038
			72	0.038
			75	0.039
			77	0.039
37.2			78	0.041
37.2			81	0.043
			84	0.044
			87	0.0445
			90	0.045
39.75			94	0.047
42.5			97	0.048
			98	0.049[d,f]

T. & P. CROSSING

(c) PILES T4–T5; TWO, 50-FT PILES

Total load per pile (tons) (1)	Time, in hr (2)	Settlement, in ft (3)
0	0	0
15	0	0.001
	2	0.005
	4	0.005
20	4	0.006
	6	0.006
	8	0.0065
25	8	0.009
	12	0.009
30	12	0.012
	16	0.012
35	16	0.014
	18	0.014
	20	0.015
40	20	0.015
	24	0.015
45	24	0.021
	28	0.021
50	28	0.021
	30	0.021
	32	0.022
55	32	0.024
	34	0.024
	36	0.026
60	36	0.030
	38	0.033
	40	0.034
	42	0.034
	44	0.036[d]

(d) PILES T14–T17; FOUR, 50-FT PILES

Total load per pile (tons) (1)	Time, in hr (2)	Settlement, in ft (3)
0	0	0
20	2	0.010
	6	0.010
	8	0.011
25	8	0.013
	10	0.014
	12	0.015
	14	0.016
30	14	0.019
	16	0.021
	18	0.022
	20	0.024
35	20	0.025
	22	0.026
	24	0.027
	26	0.028
40	26	0.029
	28	0.029
	30	0.030
	32	0.0305
45	32	0.034
	34	0.035
	36	0.0355
	38	0.036
50	38	....[e]

(e) PILES T20–T27; EIGHT, 50-FT PILES

Total load per pile (tons) (1)	Time, in hr (2)	Settlement, in ft (3)
0	0	0
22.5	7:30	0.012
	10:30	0.015
	13:30	0.0175
	16:30	0.0175
	19:30	0.018
26.25	19:30	0.019
	22:30	0.019
	25:30	0.0205
	28:30	0.021
	31:30	0.021
30.0	31:30	0.023
	37:30	0.023
	40:30	0.025
	43:30	0.025
33.75	43:30	0.027
	46:30	0.027
	49:30	0.032
	55:30	0.032
37.5	55:30	0.034
	58:30	0.034
	61:30	0.038
	64:30	0.041
	67:30	0.041
40	67:30	0.0435
	70:30	0.044
	73:30	0.0455
	79:30	0.0455
	82:30	0.047
	85:30	0.0485
	88:30	0.049
	91:00	0.0495

[a] Data for groups T37–T52 and retest group T11–T19, plotted in Figs. 2 and 3, are not included in Table 3. [b] After 243 days, no appreciable increase in settlement was observed. [c] The group was leaning, settling 0.082 ft toward a corner having the piles with the smallest diameters. [d] Load was now removed because of the steepness of the settlement curve. [e] Group failed by tipping and breaking piles. [f] After unloading, the group had an average permanent settlement of 0.042 ft.

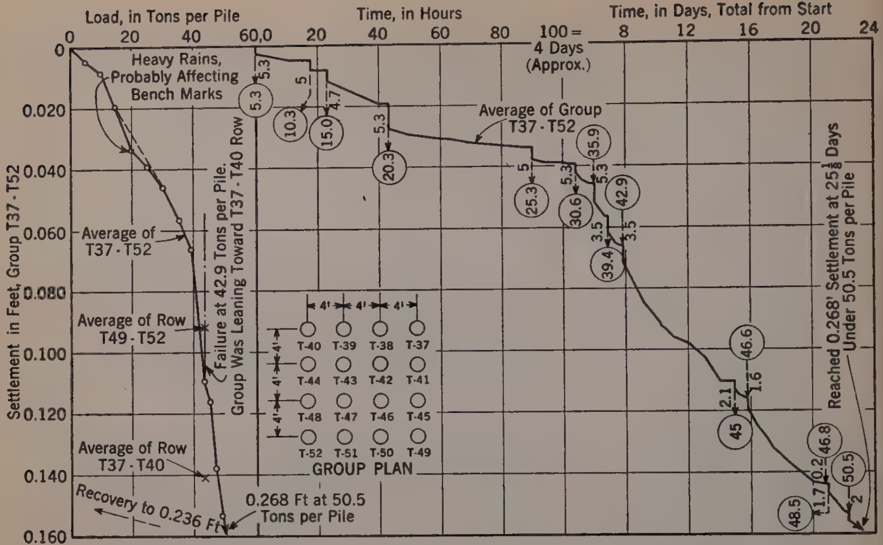


FIG. 3.—SIXTEEN-PILE GROUP T37–T52; 60-FT PILES; N.O.T. & M. CROSSING

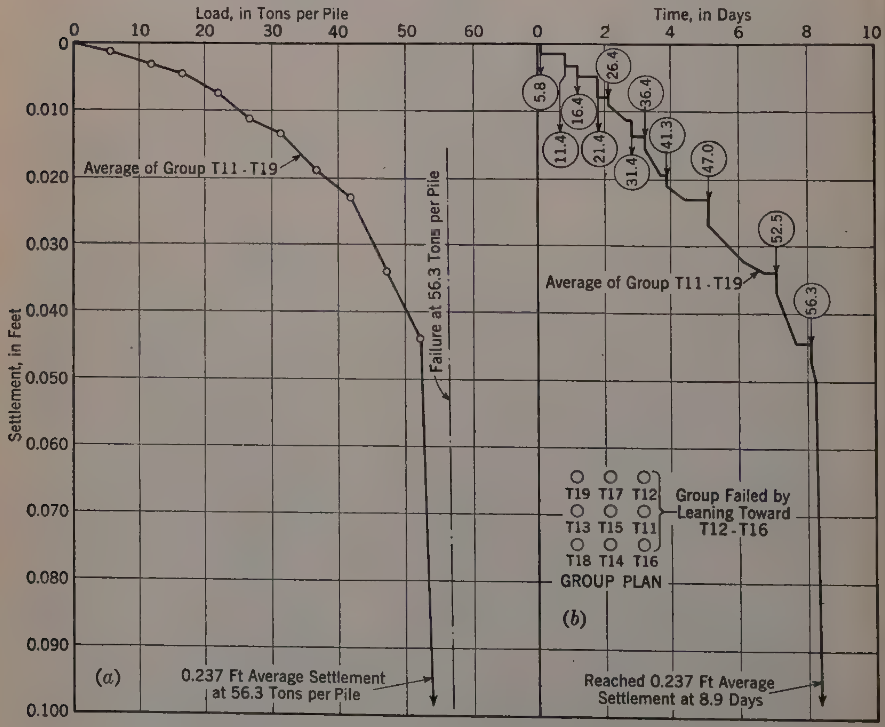


FIG. 4.—RETEST—9-PILE GROUP T11–T19; 60-FT PILES; N.O.T. & M. CROSSING

As illustrated in Fig. 2, the test loads were applied in the form of long, narrow, concrete blocks completely spanning the group and laid crosswise in successive layers. They were applied in increments until failure had been reached as indicated by "plunging" or by a continuation of settlement with time. These concrete blocks were supplemented in the case of the 16-pile group by additional loads in the form of water tanks. Settlements were measured accurately from bench marks in the ground surface that were far enough from the groups not to be influenced by the loads.

The results of all tests, to failure, of timber piles plotted both with relation to the load and settlement, as well as to the time and settlement, are given in Tables 2 and 3 and in Figs. 3 and 4. In these figures, the numbers in circles are the total loads per pile, in tons. The numbers on the adjoining arrows are load increments. These tests all demonstrate that larger groups fail more slowly than smaller groups and that individual piles fail by plunging. Table 4 gives the skin friction values at failure for each of the timber friction piles loaded to failure, and for each of the concrete friction piles loaded to failure.

TABLE 4.—SKIN FRICTIONS, INDIVIDUAL PILES

(a) TAPERED TIMBER PILES				(b) CONCRETE PILES, PARALLEL SIDES		
Crossing	Pile No.	Length (ft)	Skin friction[a]	Crossing	Pile No.	Skin friction[a]
N.O.T. & M.	T1	52	685	N.O.T. & M.	C1	420
N.O.T. & M.	T10	59.5	900	N.O.T. & M.	C2	380
N.O.T. & M.	T20	85	680	T. & P.	C1	604
N.O.T. & M.	T34	83	777	T. & P.	C2	702
T. & P.	T18	50.5	847	T. & P.	C3	562
Average	..	....	778	T. & P.	C9	551
				Port Allen	C22	688
				Average	..	558

[a] In pounds per square foot of embedded area based on the average diameter.

The parallel-sided concrete piles, when loaded to failure, did not develop the same supporting capacity per unit of embedded area as did tapered timber piles, although one would assume that the frictional resistance of the surface of a concrete pile should be at least equal to the frictional resistance of the smooth surface of a timber pile. This may be due either to the wedging action of the tapered timber pile or to some very favorable stress distribution along its sides due to the taper. The average skin friction of all tapered timber piles loaded to failure in this series of tests was 778 lb per sq ft of embedded area, whereas the average for the parallel-sided concrete piles was only 558 lb per sq ft.

NOTATION

The following notation is used in this paper:

L = length of pile as driven;
L_1 = depth from pile tip to point in ground below tip;
P = applied load; P' = point load, or load delivered by circumferential band of height dz.
q_h = unit horizontal pressure in radial direction;
q_v = unit vertical pressure;
R = diagonal distance from point load to point in soil at which pressure or shear is desired;

R_c = ratio of load delivered per linear foot of pile at top of pile to load delivered per linear foot at bottom of pile;
r = radial distance from pile center to point in ground;
s_v = unit vertical shear perpendicular to r;
s_x = unit horizontal shear parallel to x;
Z = depth from ground surface to point in ground above pile tip;
Z_1 = depth from ground surface to point in ground below pile tip.

THEORY OF FRICTION PILE BEHAVIOR

Various theories and formulas for the distribution of the load delivered by a pile to the soil were considered in this study. An attempt was made to apply some simple rule for the distribution of pile loads like the straight-line distribution at 60° to the horizontal that is common in some specifications; but this series of tests when plotted would not support any straight-line treatment, and pointed to the necessity for a more complex treatment; nor did the test results appear to check the formulas proposed by R. D. Mindlin,[2] Assoc. M. Am. Soc. C. E., which prescribe tension in the soil for loads applied beneath the ground surface. Only the classic Boussinesq formulas for the distribution of pressure through homogeneous material beneath a point load, when expanded for application to a line load, were found to check with the test results. If further tests are made, they may reveal that some simple and yet reasonably approximate

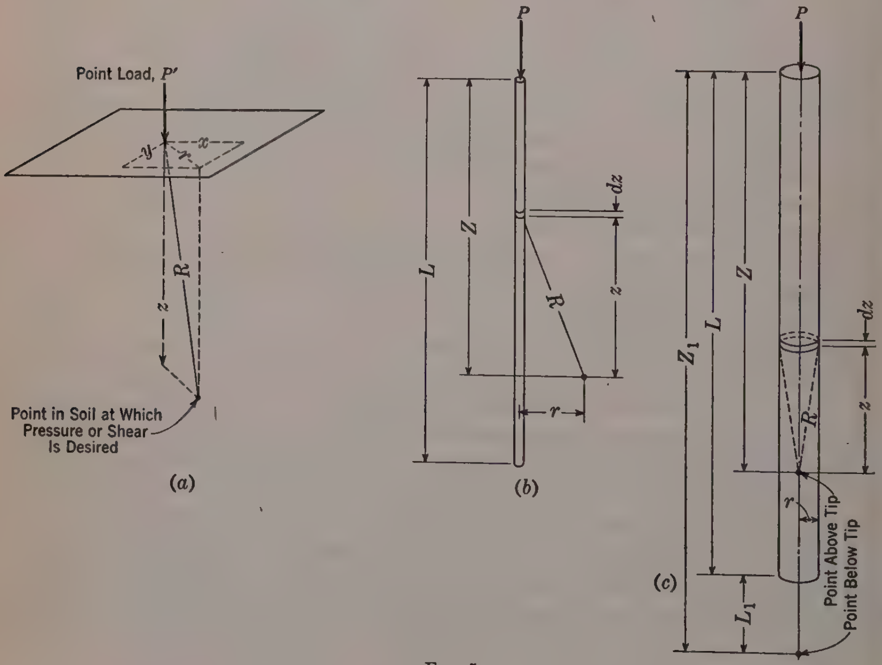


FIG. 5.

---

[2] "Force at a Point in the Interior of a Semi-Infinite Solid," by Raymond D. Mindlin, *Physics*, May, 1936.

theory may be applicable. The Boussinesq point-load formulas, which have been used frequently to determine pressures beneath buried foundations, and which were made the basis of the following analysis, are as follows (see Fig. 5(*a*)):

The unit vertical pressure (parallel to the *z*-axis) is

$$q_v = \frac{3\,P'\,z^3}{2\,\pi\,R^5} \qquad \text{(1a)}$$

The unit horizontal pressure, parallel to *r* (if Poisson's ratio for the soil is 0.5), is

$$q_h = \frac{3\,P'\,r^2\,z}{2\,\pi\,R^5} \qquad \text{(1b)}$$

The unit vertical shear, perpendicular to *r*, is

$$s_v = \frac{3\,P'\,r\,z^2}{2\,\pi\,R^5} \qquad \text{(1c)}$$

The unit horizontal shear, parallel to *x*, is

$$s_x = \frac{3\,P'\,x\,z^2}{2\,\pi\,R^5} \qquad \text{(1d)}$$

The unit horizontal shear, parallel to *r*, is numerically equal to that in Eq. 1*c*.

A friction pile delivers to the soil a vertical line of load through vertical shearing stresses, developed at the surface of the pile, to result in bearing pressures beneath the pile and shearing stresses around it. It was, therefore, necessary to expand Boussinesq's point-load formulas for pressures and shears by integration to apply to such a vertical line of load. These integrations follow:

Vertical Shear at Radius "r" from Pile.—If the load delivered to the soil per linear foot of pile varies from some value at the tip to R_c times that value at the butt, the load delivered by any circumferential band of height *dz* (see Fig. 5(*b*)) may be regarded as a point load having the value:

$$P' = \left[1 + \frac{z + L - Z}{L}\,(R_c - 1)\right]\frac{2\,P\,dz}{L\,(R_c + 1)}$$

$$= \left[\frac{(L\,R_c - Z\,R_c + Z) + z\,(R_c - 1)}{L}\right]\frac{2\,P\,dz}{L\,(R_c + 1)} \qquad \text{(2a)}$$

Based on the Boussinesq formula,

$$s_v = \frac{3\,P'\,r\,z^2}{2\,\pi\,(r^2 + z^2)^{2.5}} \qquad \text{(2b)}$$

the vertical shear at a point above the tip at depth *Z* is:

$$s_v = \int_0^z \frac{3\,r\,z^2}{2\,\pi\,(r^2 + Z^2)^{2.5}}\left[\frac{(L\,R_c - Z\,R_c + Z) + z\,(R_c - 1)}{L}\right]$$

$$\times \frac{2\,P\,dz}{L\,(R_c + 1)} = \frac{P\,r}{\pi\,L^2\,(R_c + 1)}\left[\frac{(L\,R_c - Z\,R_c + Z)Z^3}{r^2\,(r^2 + Z^2)^{1.5}}\right.$$

$$\left. + (R_c - 1)\left(\frac{2}{r} - \frac{Z^2}{(r^2 - Z^2)^{1.5}} - \frac{2}{(r^2 + Z^2)^{0.5}}\right)\right] \qquad \text{(2c)}$$

Vertical Pressure at Radius "r," or on Axis of Pile of Radius "r."—If the load delivered to the soil per linear foot of pile varies from some value at the tip to R_c times that value at the butt, the load delivered by any circumferential band of height dz (see Fig. 5(*c*)):

$$P' = \left[1 + \frac{z + L - Z}{L}(R_c - 1)\right]\frac{2\,P\,dz}{L\,(R_c + 1)}$$

$$= \left[\frac{(L\,R_c - Z\,R_c + Z) + z\,(R_c - 1)}{L}\right]\frac{2\,P\,dz}{L\,(R_c + 1)} \quad \text{.......(3a)}$$

Based on the Boussinesq formula,

$$q_v = \frac{3\,P'\,z^3}{2\,\pi\,(r^2 + z^2)^{2.5}} \quad \text{.......................(3b)}$$

the vertical pressure at a point above the tip at depth Z is

$$q_v = \int_0^Z \frac{3}{2\,\pi} \times \frac{z^3}{(z^2 + r^2)^{2.5}} \times \frac{(L\,R_c - Z\,R_c + Z) + z\,(R_c - 1)}{L}$$

$$\times \frac{2\,P\,dz}{L\,(R_c + 1)} = \frac{3\,P}{\pi\,L^2\,(R_c + 1)}\left\{-\frac{(L\,R_c - Z\,R_c + Z)\,(3\,Z^2 + 2\,r^2)}{3\,(r^2 + Z^2)^{1.5}}\right.$$

$$+ \frac{2\,(L\,R_c - Z\,R_c + Z)}{3\,r} + (R_c - 1)\left[-\frac{Z^3}{3\,(Z^2 + r^2)^{1.5}}\right.$$

$$\left.\left. - \frac{Z}{(Z^2 + r^2)^{0.5}} + \log_e\left(\frac{Z + (Z^2 + r^2)^{0.5}}{r}\right)\right]\right\} \quad \text{............(3c)}$$

The Vertical Pressure at Any Point Below the Tip at Depth Z_1.—The load delivered by any circumferential band of height dz:

$$P' = \left[1 + \frac{(z - L_1)}{L}(R_c - 1)\right]\frac{2\,P\,dz}{L\,(R_c + 1)}$$

$$= \left[\frac{(Z_1 - R_c\,L_1) + z\,(R_c - 1)}{L}\right] \times \frac{2\,P\,dz}{L\,(R_c + 1)} \quad \text{..........(4a)}$$

Applying the limits of Z_1 and L_1, the vertical pressure at a point below the tip is expressed by

$$q_v = \frac{3\,P}{\pi\,L^2\,(R_c + 1)}\left\{\frac{-\,(Z_1 - R_c\,L_1)\,(3\,Z^2_1 + 2\,r^2)}{3\,(r^2 + Z^2_1)^{1.5}} + R_c - 1\right.$$

$$\times \left[\frac{-\,Z^3_1}{3\,(Z^2_1 + r^2)^{1.5}} - \frac{Z_1}{(Z^2_1 + r^2)^{0.5}} + \log_e\left(\frac{Z_1 + (Z^2_1 + r^2)^{0.5}}{r}\right)\right]$$

$$+ \frac{(Z_1 - R_c\,L_1)\,(3\,L^2_1 + 2\,r^2)}{3\,(r^2 + L^2_1)^{1.5}} - (R_c - 1)\left[\frac{-\,L^3_1}{3\,(L^2_1 + r^2)^{1.5}}\right.$$

$$\left.\left. - \frac{L_1}{(L^2_1 + r^2)^{0.5}} + \log_e\left(\frac{L_1 + (L^2_1 + r^2)^{0.5}}{r}\right)\right]\right\} \quad \text{..........(4b)}$$

For pressures on the pile axis, use a value of r equal to the radius of the pile at, or immediately above, the point in the ground. Precisely the same formulas determine the vertical pressures around a pile, assuming it to be a vertical line of load, and setting r = radial distance from the pile axis.

The ratio "R_c" introduced in these integrations produces flexibility in the theory. As applied to timber piles, it takes account of the natural pile taper

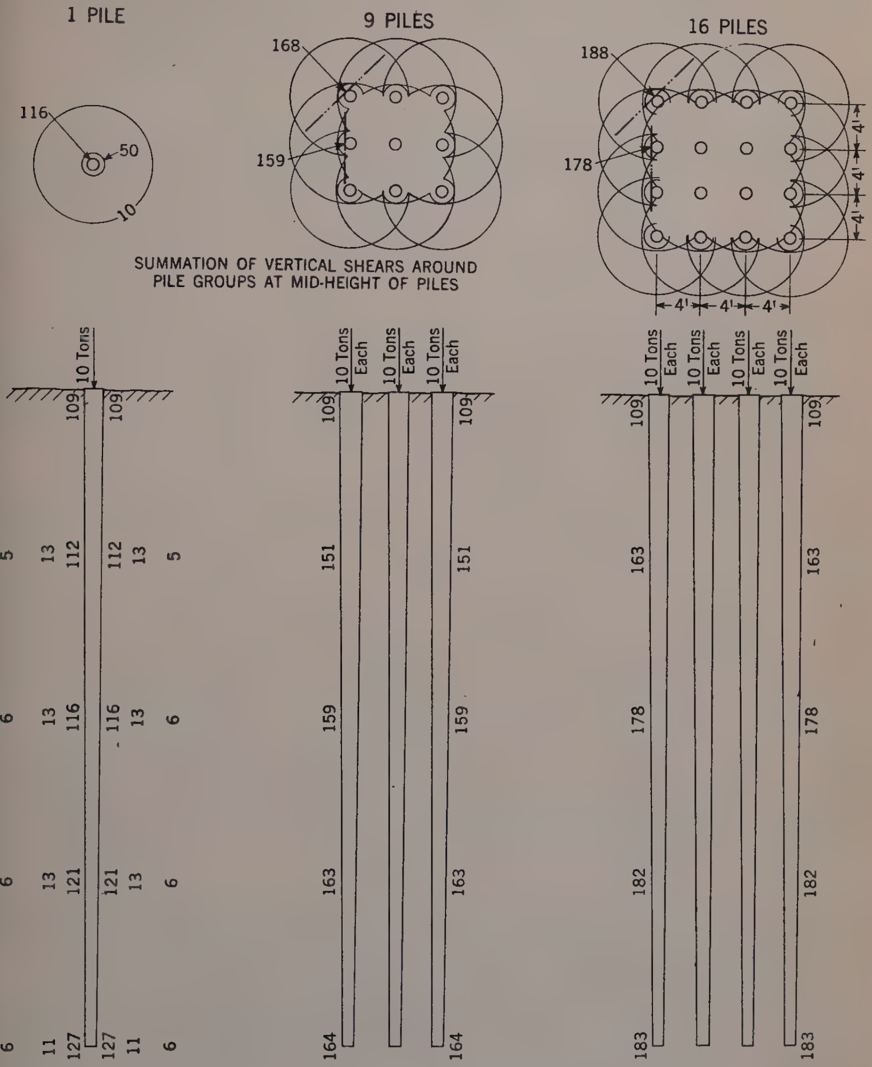


FIG. 6.—VERTICAL SHEARS (IN POUNDS PER SQUARE FOOT) UNDER LOADS OF 10 TONS PER PILE (L = 60 Ft; Diameters at Butt and Tip, 14 In. and 8 In.; 4-Ft Spacing; and R_c = 1.5)

as well as the internal friction of the soil. When an untreated timber friction pile is withdrawn from soil after having been loaded, it is generally coated with a thin film of the material, indicating excellent adhesion between the pile and the soil, and showing that failure occurred by shearing the soil. If the shearing strength of the soil were constant from butt to tip, the load delivered by shear to the soil per linear foot of pile would vary with the diameter and would be greater at the butt than at the tip. Since the shearing strength of the soil is slightly increased from butt to tip, however, as a result of a build-up of internal friction due to greater horizontal earth pressures toward the tip, this ratio of butt load to tip load per linear foot of pile will be somewhat less than the ratios of the diameters, and may be represented by the factor R_c, which appears in the pile formulas. This factor is a function of pile length, pile taper, and internal friction of the soil.

The shears existing around a single pile, a 9-pile group, and a 16-pile group, using the derived formulas for vertical shears, are presented graphically in Fig. 6. The shears are based on the assumption of $R_c = 1.5$, which value substantially checks the test results for timber piles. The computations were facilitated by the use of the curves of Fig. 7, in which the general shear formula (Eq. 2c) is plotted for various depths around a pile of unit length and under a

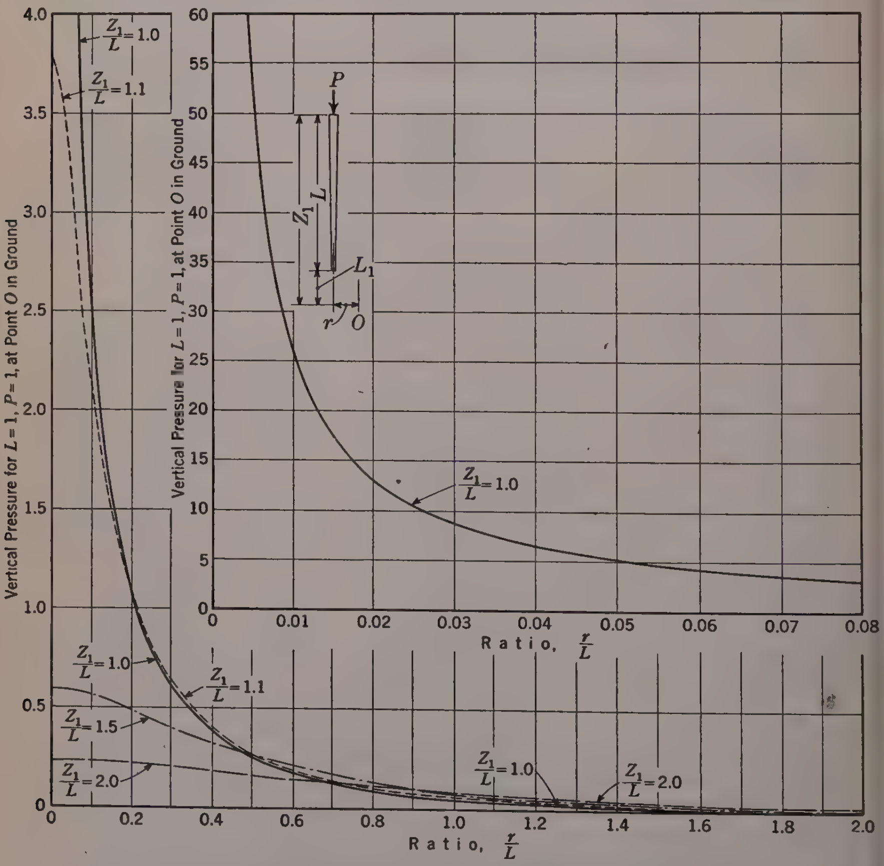


FIG. 7.—UNIT VERTICAL SHEARS AROUND A SINGLE FRICTION PILE ($L = 1$; $P = 1$; $R_c = 1.5$)

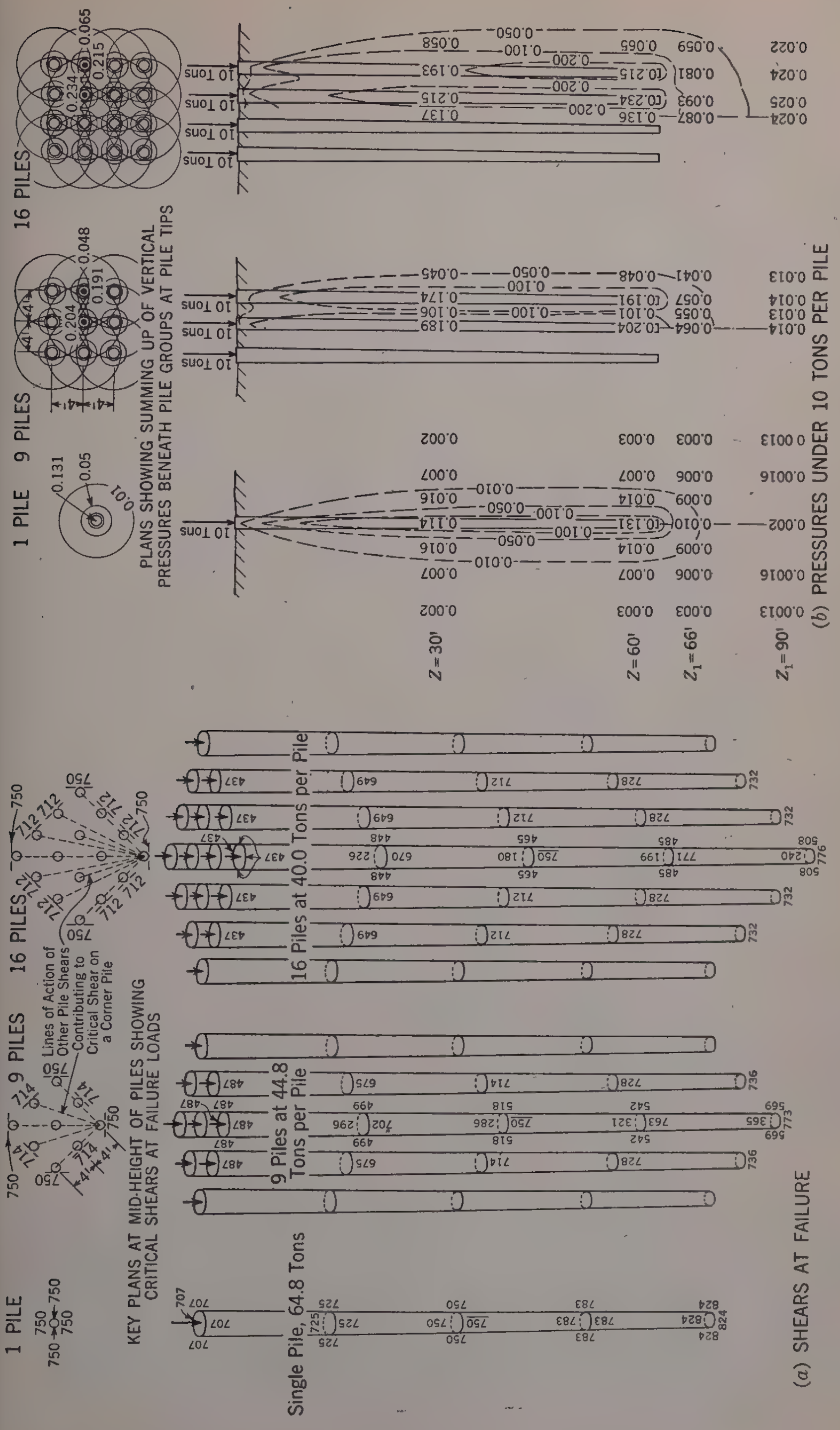

1 PILE
9 PILES
16 PILES
Lines of Action of Other Pile Shears Contributing to Critical Shear on a Corner Pile
KEY PLANS AT MID-HEIGHT OF PILES SHOWING CRITICAL SHEARS AT FAILURE LOADS
Single Pile, 64.8 Tons
9 Piles at 44.8 Tons per Pile
16 Piles at 40.0 Tons per Pile
(a) SHEARS AT FAILURE
PLANS SHOWING SUMMING UP OF VERTICAL PRESSURES BENEATH PILE GROUPS AT PILE TIPS
10 Tons
Z = 30'
Z = 60'
Z₁ = 66'
Z₁ = 90'
(b) PRESSURES UNDER 10 TONS PER PILE

FIG. 8.—VERTICAL SHEARS AND PRESSURES AROUND 60-FT PILES (Diameters at Butt and Tip, 14 In. and 8 In.; 4-Ft Spacing; and $R_e = 1.5$)

load of one unit. To find the shear around a pile of any length L, under a load P, multiply the vertical shear from these curves by the factor $\frac{P}{L^2}$. Fig. 6 shows that the vertical shear effects overlap when piles are placed near each other, and result in maximum shear stresses on the outer surfaces of the perimeter piles in a group, which are greatest at the corner piles. Hence, failure is likely to occur by the development of excessive vertical shears outside the edge or corner piles. This was indicated by the Morganza tests when a group failed by leaning toward a side or corner.

Consequently, it is a basic assumption of the design method described in this paper that small friction pile groups fail when the corner or edge piles break loose by shear failure from the bulbs of pressure that would otherwise distribute the pile loads safely. Larger groups than those tested may settle as a result of consolidation due to vertical pressures under the center of the group and may be investigated for this condition by the formulas for vertical pressure (Eqs. 3*c* and 4*b*). However, in computing the load-carrying capacity of any arrangement of piles, the group should first be investigated against failure in shear along the outer piles.

For the purpose of illustrating the contribution of shear from each pile in a group to the shear on the outer piles, and the further fact that the shear at midheight of a corner pile of a group is most critical, the shearing stresses under failure loads are shown in Fig. 8(*a*) around a single pile, a group of nine piles, and a group of sixteen piles, computing these shearing stresses at the butt, the tip, midheight, and the two quarter points, and using $R_c = 1.5$. In the computations of the failure loads on the piles in these various groups, an average shearing strength of 750 lb per sq ft at failure of a single pile was used for convenience of computation. It will be noted that the shearing stresses, in pounds per square foot of embedded surface on the single pile, increase slowly from butt to tip, being larger toward the tip because of the better shear value of the soil there due to internal friction. It will also be noted on the 9-pile and 16-pile groups of Fig. 8(*a*) that the vertical shears shown around the outside piles in no case exceed the vertical shears at corresponding heights on a single pile, and that, as the number of piles in the group increases, the load per pile must decrease in order to satisfy this condition, owing to the contributions of all other piles to the shears on each outside pile. The critical point around the perimeter for groups of this size is seen to be at midheight of the outside surface of the corner pile, and, if the assumed shearing value of 750 lb per sq ft at the midheight of this pile is exceeded, failure will occur and spread progressively to the less critically stressed points around the group perimeter.

DETERMINATION OF SHEAR-FAILURE LOAD OF A GROUP

A method of determining the shear-failure load of a group of piles, using the derived formula (Eq. 2*c*), would consist, first, of loading a single pile to failure, as determined by the load at which the pile settlement under small additional increments of load would increase rapidly and the pile would no longer come to rest. This test load divided by the contact area is the average skin friction, which equals the shear-failure value at midheight of a pile. A group is assumed to fail by shear at this test-determined value per square foot along a plane intersecting the outer surface of a corner pile, and drawn approximately normal to the line connecting this corner pile with the center of gravity of the group. A load P is assumed on each pile of the group arrangement to be investigated. The contribution of each pile in the group to the midheight shear of the corner or critical pile is found by using the general shear formula (Eq. 2c) and multiplying the contribution of each of the other piles by the sine of the angle of action with the diagonal plane of failure. When these contributions from all piles in terms of a load P per pile are added to the usual midheight shear on the surface of the corner pile (which is equal to P divided by its penetrated contact area), a decimal number is obtained, multiplied by P. The shear-failure value per square foot is divided by this decimal number, and the quotient is the failure load P per pile.

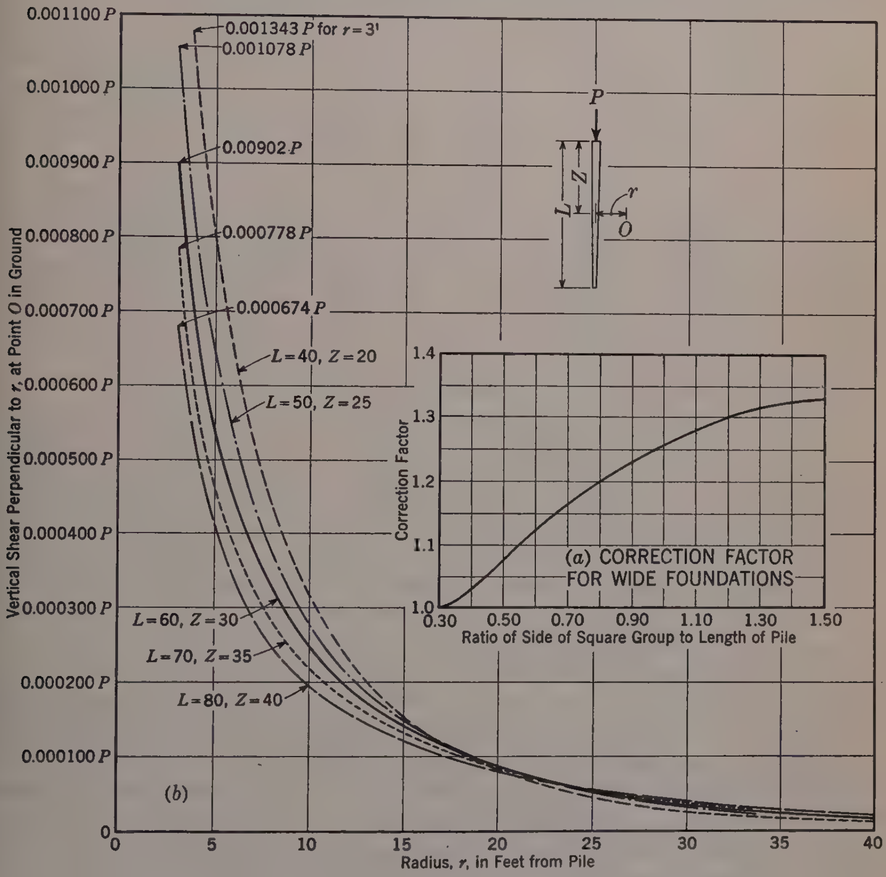


FIG. 9.—VERTICAL SHEARS AROUND FRICTION PILES, AT MIDHEIGHT (R_c = 1.5)

To facilitate the computation, curves have been drawn in Fig. 9, plotting the shear formula for $R_c = 1.5$ for various actual pile lengths from 40 ft to 80 ft, giving the shear at midheight of a pile at any radius from the pile. From these curves it is possible to read directly the fractional part of P contributed as shear by any pile in the group to the shear on the critical pile at the corner, on a plane normal to the line connecting the two piles.

In order to demonstrate the accuracy of the use of these derived formulas, Table 5 has been compiled to show the comparison of theoretical and actual failure loads for the two 9-pile test groups, T11–T19 and T21–T29, and for the 16-pile test group, T37–T52. The theoretical failure load was corrected in each case for a small amount of eccentricity in the pile group resulting from a large number of smaller diameter piles driven toward one corner or side. Table 5 also shows the same comparison for the 8-pile test group (N.O.T. & M.), T2–T9; 8-pile test group (T. & P.), T20–T27; the 4-pile test group (T. & P.) T14–T17; and 2-pile test group (T. & P.), T4 and T5.

TABLE 5.—Comparison of Theoretical and Actual Failure Loads per Pile, Morganza Timber Test Pile Groups

Crossing	Pile No.	Piles per group	Shape	Pile spacing, in ft	Length of pile, in ft	Diameter of pile, in in.[a]	Bending effect[b]	Shear Value Pertaining to Group		Failure Loads, in Tons per Pile	
								Lb per sq ft	Pile used as basis	Theoretical	Actual
N.O.T. & M.	T21–T29	9	One square	4	60	14.265	1.022	778	Average	54	55.3
N.O.T. & M.	T11–T19	9	One square	4	60	14.31	1.0243	778	Average	54	56.3
N.O.T. & M.	T37–T52	16	One square	4	60	12.93	1.065	778	Average	43	42.9
N.O.T. & M.	T2–T9	8	Two squares[c]	3	50.5	13.55	1.000	685	T1	41.5	42.5
T. & P.	T20–T27	8	Rectangle	3	50	12.00	1.000	778	Average	39	40
T. & P.	T14–T17	4	Square	3	50	12.56	1.076	847	T18	47	50
T. & P.	T4–T5	2	Rectangle	3	50	12.72	1.000	847	T18	60	60

[a] Over-all average diameter. [b] Relative effect of the eccentricity of the center of gravity of the piles. [c] Separated by 9 ft.

Unfortunately individual piles were not driven and tested to failure adjacent to all of the groups because time and available funds were both limited, and, in computing the carrying capacity of those groups where no single pile test was made nearby, the average shear value of 778 lb per sq ft of all single piles tested to failure was used. On the other hand, where single piles had been driven and tested to failure near some of the smaller groups, the actual shear values of these single piles were used in computing the theoretical failure load. It will be noted that theoretical failure loads computed on this basis checked very closely with the actual failure loads.

Modifications of Method for Large Groups

Since each pile distributes its load outward and downward, it is evident that, as the group size increases, the point of critical shear on the outer piles will tend to move downward. From computations of various sizes and arrangements of groups, it has been found that, where the side of a square group is more than one third of the length of the piles, the critical shear on the corner piles occurs lower than the midheight of the pile. Thus, the failure load determined from curves plotted for midheight computations (Fig. 9(*b*)) should be multiplied by the factors in Fig. 9(*a*).

COMPUTATION OF SAFE LOADS

This method primarily establishes the failure load per pile of a group in shear. In the actual design of structures supported on friction piles, the working load chosen per pile should provide ample safety factors (1) against possible shear failure, and (2) against excessive settlements from vertical pressures, giving due regard to the relative proportions of dead, live, and dynamic loads. The dead load is the constant load that may produce excessive settlement over long periods of time without necessarily resulting in a shear or total failure. It is probable that a working load, or dead plus live load, may be chosen some-

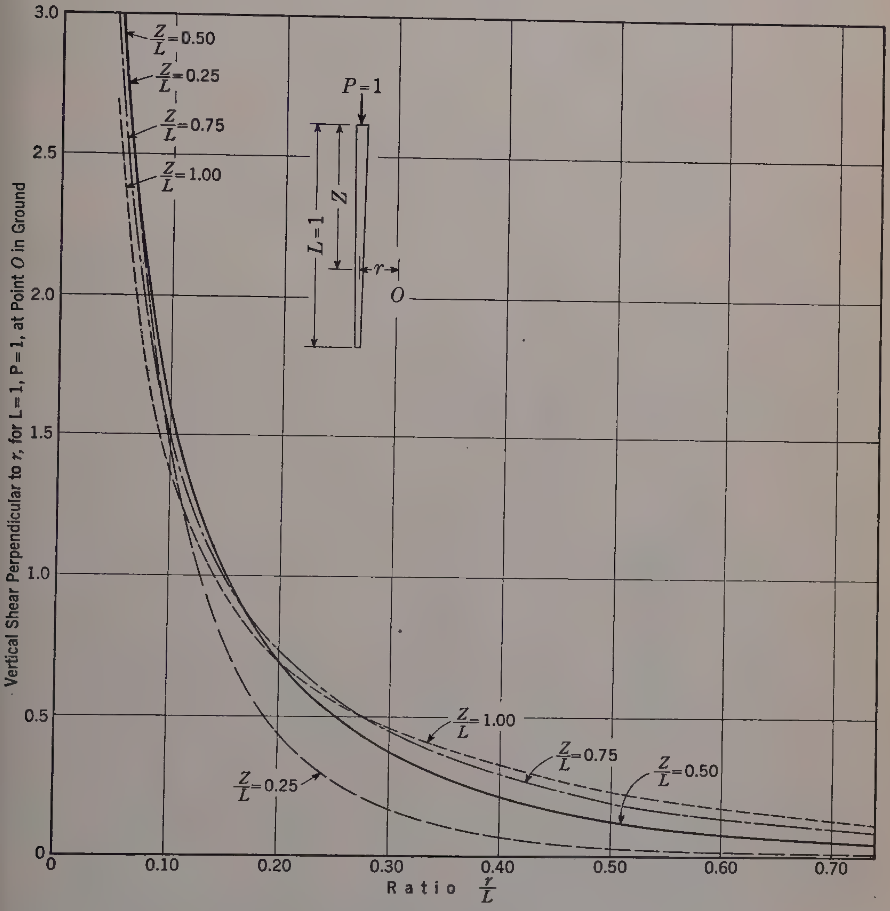


FIG. 10.—UNIT VERTICAL PRESSURES BENEATH A SINGLE FRICTION PILE ($L = 1$; $P = 1$; $R_e = 1.5$)

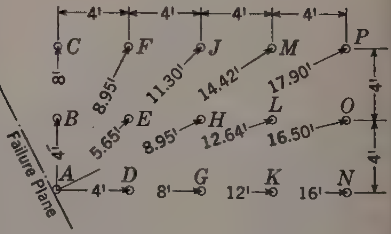
what closer to the shear-failure load in large groups than in small groups, because of the relatively sluggish failure of larger groups as illustrated by the behavior of the 16-pile group under test.

Illustrative Problem.—The following example illustrates the manner in which this pile theory is applied to an actual problem of computing the failure loads and pressures of a 15-pile group. The calculations of vertical pressures were facilitated by the use of the curves of Fig. 10, in which the vertical pressure formulas have been plotted for a pile length $L = 1$ and load $P = 1$.

In this problem, fifteen 60-ft piles, with a specified minimum tip diameter of 8 in. and a minimum butt diameter of 14 in., are laid out at 4-ft centers in a rectangle of five piles by three piles. They are to be equally loaded to 15 tons total load per pile. The average shear value of the soil at the failure of a single pile is 750 lb per sq ft.

Example 1. Load per Pile in a Vertical Shear Failure.—The total shear in piles of the group is computed as shown in Table 6. For the corner pile A, $\frac{P}{60 \times 0.92\, r} = 0.005780\, P$, the remaining values in Col. 4 being Col. 2 times Col. 3.

TABLE 6.—SHEAR ON CORNER PILE OF GROUP UNDER EQUAL LOADS PER PILE

Pile	Radius, r, in ft[a] (1)	Vertical shears[b] at $Z = 30$ ft ($R_e = 1.5$) (2)	Sine of angle with failure plane (3)	Shear on failure plane under equal loads P per pile[c] (4)	Diagram of group
A	....	....	....	0.005780	 4' 4' 4' 4' C F J M P 8' 8.95' 11.30' 14.42' 17.90' 4' B E H L O 5.65' 8.95' 12.64' 16.50' 4' Failure Plane A D G K N 4' 8' 12' 16'
B	4	0.000660	0.447	0.000295	
C	8	0.000327	0.447	0.000146	
D	4	0.000660	0.894	0.000590	
E	5.65	0.000472	0.950	0.000449	
F	8.95	0.000283	0.802	0.000227	
G	8	0.000327	0.894	0.000292	
H	8.95	0.000283	1.000	0.000283	
J	11.30	0.000205	0.950	0.000195	
K	12	0.000190	0.894	0.000170	
L	12.64	0.000175	0.991	0.000173	
M	14.42	0.000140	0.993	0.000139	
N	16	0.000125	0.894	0.000112	
O	16.50	0.000118	0.977	0.000115	
P	17.90	0.000105	1.000	0.000105	
Total group shear =				0.009071 P	

[a] Read from the inset diagram of the group. [b] Read from Fig. 9, "pile B = 0.000660 P," etc. [c] Compute: "Pile B = Col. 2 × Col. 3 = 0.000295 P," etc. (Col. 4).

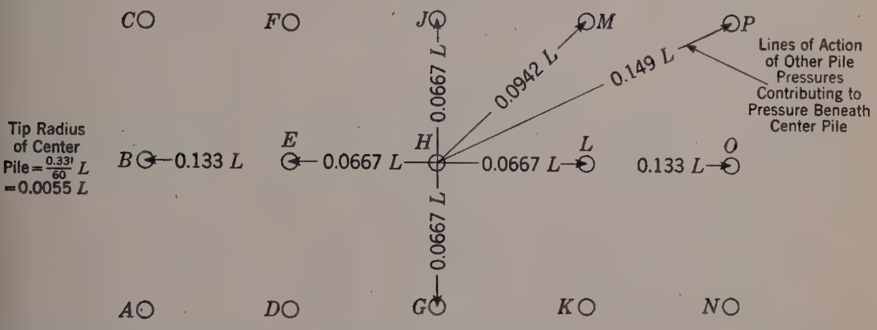
From the total group shear in Table 6, the failure load P per pile equals

$$\frac{750 \text{ lb per sq ft (failure shear)}}{0.009071} = 82{,}700 \text{ lb} = 41.4 \text{ tons.}$$

(In the inset diagram of Table 6 the failure plane is drawn normal to the line connecting the corner pile with the center of the group—a 45° plane gives almost the same results.)

Example 2. Vertical Pressures Beneath the Group Under 15 Tons per Pile.—In Table 7, the inset diagram shows the group dimensions reduced to proportional parts of L (= 60 ft), with radial distances from the center pile to each other pile. The table shows the method and results of the computation of vertical pressures.

TABLE 7.—VERTICAL PRESSURES BENEATH THE GROUP, UNDER 15 TONS PER PILE


CO
FO
JO
OM
OP
Lines of Action of Other Pile Pressures Contributing to Pressure Beneath Center Pile
0.0942 L
0.149 L
0.0667 L
Tip Radius of Center Pile = 0.33'/60 L = 0.0055 L
B
0.133 L
E
0.0667 L
H
L
0.0667 L
O
0.133 L
0.0667 L
AO
DO
GO
KO
NO

Depth Below Tip of Pile H		Values Read from Fig. 10					Total Pressure	
In ft	Ratio Z_1/L	Pile H (Tip r = 0.0055 L)	Four piles E, G, J, L (r = 0.0667 L)	Four piles D, F, K, M (r = 0.0942 L)	Four piles A, C, P, N (r = 0.149 L)	Two piles B, O (r = 0.133 L)	In units	In tons per sq ft[a]
0	1	47.10	15.50	10.72	6.16	3.56	83.04	0.347
6	1.1	3.76	10.40	8.88	6.00	3.36	32.40	0.135
30	1.5	0.59	2.32	2.27	2.12	1.08	8.38	0.035
60	2	0.23	0.92	0.92	0.88	0.46	3.41	0.014

[a] The actual pressures in last column are equal to units in preceding column multiplied by $15/60^2$.

Two important conclusions are to be noted in the solutions of Examples 1 and 2: First, that the failure load per pile, of the 15-pile group, is slightly greater than that of the approximately equivalent 16-pile square group of Fig. 8(*a*); and second, that the vertical pressures under the center pile of the 15-pile group are slightly less than those beneath the interior piles of the 16-pile square group of Fig. 8(*b*) if it is loaded to 15 tons per pile.

These conclusions suggest that tables of shear-failure load per pile, and of pressures beneath groups for square groups of many dimensions, may be a useful guide to design. Tables 8 and 9 have been prepared with that purpose in mind. Table 8 has been prepared for a shear-failure value of 750 lb per sq ft from which results may be proportioned for other values; and Table 9 has been prepared for 10 tons per pile, from which results may be proportioned for other loads. The tables should be used subject to the following rules:

(1) A rectangular or circular group will fail in shear at a larger load per pile than a square group containing the same number of piles.

(2) A rectangular group will have smaller vertical pressures and a circular group will have larger vertical pressures than a square group containing the same number of piles.

TABLE 8.—SHEAR FAILURE LOADS PER PILE, IN TONS, FOR SQUARE GROUPS OF PILES OF VARIOUS LENGTHS AND SPACINGS

(R_c = 1.5; Shear Value of Soil = 750 Lb per Sq Ft)

No. of rows square	No. of piles	2.5-FT SPACING			3-FT SPACING			4-FT SPACING		
		40-ft piles	60-ft piles	80-ft piles	40-ft piles	60-ft piles	80-ft piles	40-ft piles	60-ft piles	80-ft piles
Diameter, in Inches: Butt		14	14	14	14	14	14	14	14	14
Tip		9	8	7	9	8	7	9	8	7
1	1	45.2[a]	64.8[a]	82.7[a]	45.2[a]	64.8[a]	82.7[a]	45.2[a]	64.8[a]	82.7[a]
2	4	32.0[a]	46.5[a]	60.0[a]	33.6[a]	48.9[a]	62.8[a]	36.0[a]	52.3[a]	66.8[a]
3	9	25.7[a]	37.2[a]	48.0[a]	27.8[a]	40.1[a]	51.8[a]	31.2[a]	44.8[a]	57.3[a]
4	16	21.9[a]	31.4[a]	40.6[a]	24.5[a]	34.7[a]	44.5[a]	28.5[a]	40.0[a]	50.9[a]
5	25	19.5[a]	27.5[a]	35.4[a]	22.3[a]	31.0[a]	39.6[a]	26.0	36.8[a]	46.2[a]
6	36	17.9[a]	24.9[a]	31.7[a]	20.4	28.4[a]	35.8[a]	23.9	34.7[a]	43.0[a]
7	49	16.5	22.9[a]	28.8[a]	18.6	26.5[a]	33.0[a]	22.4	32.1	40.4[a]
8	64	15.3	21.4[a]	26.7[a]	17.5	24.9	30.9[a]	21.3	30.1	38.1
9	81	14.4	20.2[a]	24.9[a]	16.4	23.1	29.1[a]	20.2	28.4	35.8
10	100	13.5	18.9	23.5[a]	15.9	21.7	27.4	19.6	27.2	33.9
11	121	12.9	17.8	22.3[a]	15.1	20.7	26.1	19.0	25.9	32.4
12	144	12.3	17.0	21.1	14.5	19.9	24.7	18.5	25.0	31.1
13	169	11.9	16.2	20.1	14.2	19.1	23.6	18.2	24.3	30.0
14	196	11.7	15.5	19.1	13.8	18.4	22.5	17.8	23.6	28.9
15	225	11.5	15.1	18.1	13.5	17.9	21.7	17.7	22.9	28.3
16	256	11.2	14.6	17.5	13.3	17.4	21.2	17.5	22.5	27.5
17	289	10.9	14.2	17.0	13.1	16.9	20.3	17.4	22.2	27.0
18	324	10.7	13.7	16.3	13.0	16.6	19.8	17.3	21.9	26.3
19	361	10.5	13.5	15.8	12.9	16.2	19.4	17.2	21.5	25.9
20	400	10.4	13.2	15.3	12.7	15.9	18.7	17.2	21.3	25.5
21	441	10.3	13.0	14.9	12.7	15.6	18.4	17.1	21.2	25.1
22	484	10.3	12.6	14.6	12.6	15.5	18.0	17.0	21.0	24.8

[a] Governed by shear summation at midheight of corner pile; all others governed by shear summation at some point below midheight of corner pile.

PRACTICAL APPLICATION OF THIS METHOD TO DESIGN

On the basis of the preceding data, the following would be the design procedure:

(1) Make sufficient borings at the site to establish the approximate uniformity of the soils over the entire area being considered, and to establish further the fact that a firm stratum for the support of point bearing piles is not economically accessible.

(2) Drive a test pile of about the length contemplated to be used in the design, and if the number of hammer blows per foot increases only gradually with increased penetration, the pile will be a friction pile, distributing load approximately in the manner suggested by this pile theory. More than one test pile may be used to sample the area thoroughly.

(3) Load the test pile, by increments, to failure, which will be indicated by a sharp increase in settlement that does not cease with time, and which should result in at least several inches of settlement. From this failure load, the shear-failure value, in pounds per square foot, may be computed by dividing by the penetrated contact area.

(4) With this shear value in mind determine, by an examination of Table 8, approximately what the failure load per pile will be for the approximate number of piles required, and select a working load per pile that provides an ample safety factor consistent with the group size and the loading characteristics. For an unusual group, the determination of failure load may be made more precisely by the method given in Example 1.

TABLE 9.—VERTICAL PRESSURES, IN TONS PER SQUARE FOOT, BENEATH CENTER PILE OF VARIOUS SQUARE PILE GROUPS LOADED TO 10 TONS PER PILE

(R_c = 1.5; 40-Ft Piles, 14-In. Butt, 9-In. Tip; 60-Ft Piles, 14-In. Butt, 8-In. Tip)

Depth, in ft, below tip	One pile	3 rows = 9 piles	4 rows = 16 piles	5 rows = 25 piles	7 rows = 49 piles	9 rows = 81 piles	11 rows = 121 piles	13 rows = 169 piles	15 rows = 225 piles	17 rows = 289 piles
(a) 40-FT PILES, 3-FT SPACING										
0	0.172	0.316	0.376	0.448	0.562	0.663	0.752	0.826	0.890	0.940
4	0.023	0.139	0.197	0.266	0.386	0.490	0.585	0.660	0.728	0.789
20	0.004	0.032	0.054	0.082	0.146	0.218	0.295	0.374	0.448	0.520
40	0.001	0.013	0.023	0.035	0.066	0.106	0.154	0.202	0.251	0.298
(b) 40-FT PILES, 4-FT SPACING										
0	0.172	0.278	0.318	0.367	0.443	0.505	0.555	0.593	0.625	0.649
4	0.023	0.117	0.157	0.208	0.287	0.352	0.407	0.453	0.490	0.523
20	0.004	0.031	0.051	0.076	0.131	0.187	0.245	0.299	0.348	0.392
40	0.001	0.013	0.022	0.034	0.064	0.099	0.137	0.173	0.211	0.251
(c) 60-FT PILES, 3-FT SPACING										
0	0.131	0.228	0.271	0.324	0.413	0.492	0.568	0.635	0.698	0.755
6	0.010	0.074	0.113	0.160	0.245	0.326	0.405	0.478	0.543	0.600
30	0.002	0.014	0.025	0.039	0.073	0.112	0.157	0.207	0.259	0.310
60	0.000	0.006	0.010	0.016	0.030	0.049	0.073	0.099	0.129	0.161
(d) 60-FT PILES, 4-FT SPACING										
0	0.131	0.204	0.234	0.271	0.331	0.384	0.433	0.475	0.510	0.541
6	0.010	0.064	0.093	0.129	0.190	0.245	0.295	0.340	0.379	0.413
30	0.002	0.014	0.025	0.039	0.070	0.104	0.142	0.182	0.221	0.258
60	0.000	0.006	0.010	0.016	0.030	0.048	0.070	0.094	0.119	0.144

(5) Find the approximate vertical pressures beneath the center of the group, under the working load chosen per pile, by examination of Table 9 (some interpolation may be necessary). If the pressures seem dangerous, or if the group is unusual, a more precise calculation may be made by the method given in Example 2.

(6) If consolidometer tests have been made on samples of the soil beneath and around the piles, it will be possible to compute, from the vertical pressures determined by use of the formulas and curves, a theoretical maximum settlement of the pile tops along the vertical center line of the footing under the working load. This theoretical settlement cannot be realized fully, especially in piles supporting a stiff pier base where the stiffness of the footing will restrict

settlements because of the lower pressures existing under the edges of the footing. If the area under a large building is thought of as one large footing in sections, however, the indicated center settlement may be realized.

Modification of Pressure Distribution due to Variation in Materials

The materials encountered on the Morganza Floodway are believed to distribute loads in a manner representative of most true friction pile foundations. If the materials were to grow progressively denser with increased penetration, the distribution of pile load to the soil would become greater toward the tip, of course, and the effect of this could be estimated by reducing the value of R_c in all the formulas from the value of 1.5 used in this paper. Ratio $R_c = 1$ would represent a pile with uniform load per foot, and $R_c = 0$ would represent a distribution of load increasing from zero at the butt to a maximum value at the tip.

As the materials would become denser and denser toward the tip, the reduction in pile strength due to grouping would become progressively less pronounced until, with pile tips bearing on rock, the permissible load per pile in a group would be equal to the permissible load on a single pile.

TABLE 10.—Mindlin-Ruderman Stress Coefficients for Pile Foundations

(Poisson's Ratio Assumed 0.5; All Values Negative (or Compression) Except As Noted by + in Table 10)

Ratio Z/L	Ratio r/L									
	0	0.1	0.2	0.3	0.5	0.7	1	1.5	2	3
(a) Friction-Load Stress Distribution $p_Z = P_{Friction} \times K_f/L^2$										
0.25	0.698	0.568	0.355	0.209	0.079	0.033	0.009	0.001	0.000	0.000
0.50	0.554	0.522	0.442	0.345	0.184	0.091	0.031	0.005	0.001	0.000
1.00	....	1.550	0.750	0.478	0.249	0.145	0.067	0.024	0.006	0.001
1.25	0.902	0.790	0.593	0.432	0.246	0.152	0.078	0.027	0.010	0.001
1.5	0.419	0.404	0.364	0.312	0.216	0.147	0.083	0.033	0.013	0.003
2.00	0.180	0.178	0.172	0.163	0.139	0.112	0.077	0.039	0.019	0.005
3.00	0.068	0.068	0.067	0.066	0.062	0.057	0.048	0.034	0.022	0.009
(b) Point-Load Stress Distribution $p_Z = P_{Point} \times K_p/L^2$										
0.25	+0.174	+0.166	+0.142	+0.090	+0.021	0.009	0.012	0.004	0.001	0.000
0.50	+0.755	+0.678	+0.476	+0.268	+0.034	0.032	0.024	0.062	0.029	0.000
1.00	......	0.120	0.119	0.110	0.096	0.078	0.051	0.021	0.008	0.001
1.25	3.89	2.59	1.221	0.494	0.145	0.082	0.051	0.023	0.011	0.001
1.5	1.04	0.948	0.736	0.513	0.237	0.118	0.061	0.026	0.012	0.002
2.00	0.289	0.281	0.263	0.239	0.183	0.130	0.076	0.035	0.012	0.004
3.00	0.086	0.086	0.084	0.082	0.076	0.068	0.055	0.036	0.026	0.008

Modification of Results Because of Other than Boussinesq Distribution

The Boussinesq distribution of stress is always downward from a point of load, and has been used with success in determining pressures beneath many buried foundations. Professor Mindlin's formulas[2] indicate a stress distribution upward by tension as well as downward by compression from a point load buried in the ground. D. M. Burmister, Assoc. M. Am. Soc. C. E., has presented the Mindlin formulas in usable form for pile foundations,[3] as reproduced in Table 10.

From Table 10(*a*), showing stress distribution around a pure friction pile with uniform load per linear foot, it has been possible to compute a curve of vertical shears around a pile at mid-height. In Fig. 11 this curve is shown in comparison with the corresponding curve from the general shear formula (Eq. 2*c*) derived in this paper, using $R_c = 1.5$, and with a curve from the same formula using $R_c = 1$ for the condition of uniform load per foot of pile.

The comparative tendency of the Mindlin distribution is to reduce pressures beneath the pile and increase the shears around the pile, thus throwing more load into distant piles of a group and making group action more dangerous than the Boussinesq distribution would indicate.

Since the curve for $R_c = 1.5$ under Boussinesq distribution checks the pile tests more closely than does the curve for Mindlin distribution, it would seem that timber friction piles distribute load more nearly in accordance with the Boussinesq formulas.

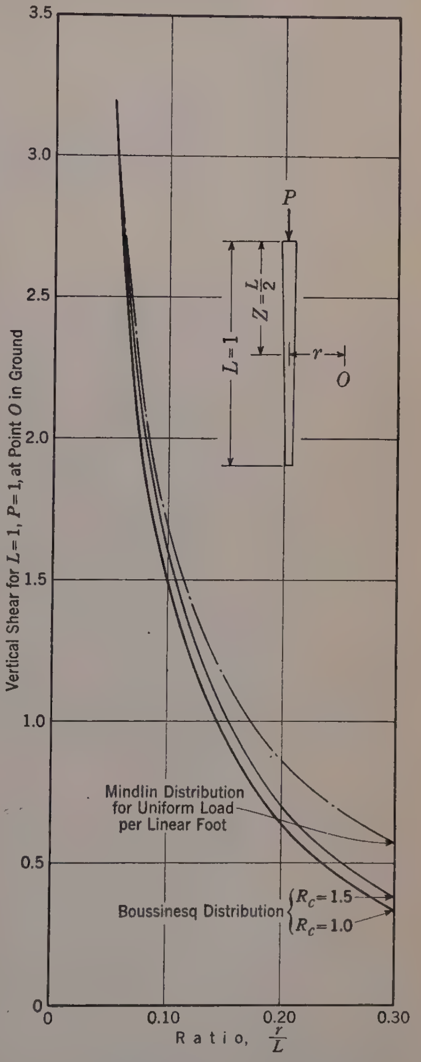


Fig. 11.—Comparison of Unit Vertical Shears Around a Single Friction Pile by Boussinesq and Mindlin Distributions (at Midheight of Pile)

IMPLICATIONS OF THIS THEORY OF FRICTION PILE BEHAVIOR

If this theory is correct in stating that shear failure occurs through the building up of critical shear stresses on the corner piles of square or rectangular groups, it would follow logically that a more efficient grouping arrangement against shear failure could be obtained by eliminating corner piles through the use of octagonal footings containing the same number of piles.

---

[3] "Proceedings of the Purdue Conference on Soil Mechanics and Its Applications," Purdue Univ., Lafayette, Ind., September 2 to 6, 1940, p. 339 (citing also "Stress Distribution Around a Loaded Pile," by J. Ruderman, Thesis No. 500 submitted to the Dept. of Civ. Eng., Columbia Univ., New York, N. Y., in 1939, in partial fulfilment of the requirements for the degree of Doctor of Science).

Where a large group of friction piles supports a stiff pier base, some redistribution of pile loads toward the edges from those calculated will take place in order that the settlement of the edge piles due to vertical pressures may become equal to the settlement of the center piles. It would be in accord with this tendency, and would distribute the pier load more equally to the individual piles, if the piles were spaced more closely in the exterior rows of the group than among the interior rows.

It would be desirable, if funds were available, to make various group arrangements and test them to failure so as to have additional test data on sizes and shapes of groups as well as the most efficient spacing. It is hoped that others interested in this subject may be able to continue the tests that were begun with this series.

Acknowledgments

The first computations using the formulas derived in this paper were presented in a report to the Corps of Engineers, U. S. Army, New Orleans District, and used in the preparation of designs for proposed railway crossings across the Morganza Floodway. The report was presented in December, 1940, by the writer's firm working with that of J. F. Coleman, Past-President and Hon. M. Am. Soc. C. E. The author is indebted to Harry J. Engel, Assoc. M. Am. Soc. C. E., for his work in the derivation of the equations used and the numerous calculations required. During the progress of the pile tests used in this article, Lt.-Col. William F. Tompkins and Lt.-Col. Clark Kittrell, Members, Am. Soc. C. E., were successive district engineers for the New Orleans District, and their cooperation, as well as that of their assistants, in making the tests is gratefully acknowledged. Roscoe Owen, M. Am. Soc. C. E., construction engineer of the N.O.T. & M. Railway Company, supervised the supplementary group pile tests and has made valuable comments upon the theory of design of timber friction pile foundations presented in this paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
PAPERS

BENDING MOMENTS IN THE WALLS OF RECTANGULAR TANKS

BY DANA YOUNG,[1] ASSOC. M. AM. SOC. C. E.

SYNOPSIS

An attempt is made herein to obtain a theoretical solution for the deflection and moments in the vertical side walls of a rectangular tank which is open at the top. More precisely, a formal analysis is made of a rectangular plate that is entirely free or unsupported at the top, perfectly fixed along the sides and bottom and subjected to a transverse hydrostatic loading. In certain cases, the walls of a rectangular tank, as shown in Fig. 1, may be considered to act as plate with these idealized edge conditions, and it is only in such cases that this analysis applies. For a tank that is square in plan and with a heavy base, perfected fixity at the sides and bottom would be closely approached. If one horizontal dimension is much greater than the other, the assumed edge conditions would not prevail. This analysis may be said to furnish the solution for an idealized case and the designer must use judgment in applying the results to a particular problem.

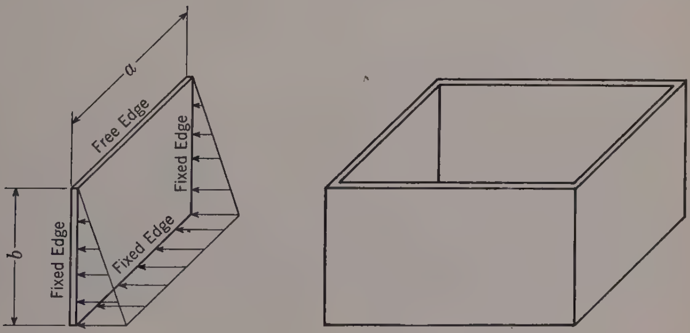


FIG. 1

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **March, 1942.**

[1] Prof. and Head, Dept. of Civ. Eng., Univ. of Connecticut, Storrs, Conn.

There are other possible applications besides that of a tank wall. The vertical slab in a counterfort retaining wall may be considered as such a plate providing the bottom edge is designed for fixity. The flat slab of a buttress dam is in the same category.

The analysis given herein is based on the so-called ordinary theory of bending of thin plates. According to this theory, the load on the plate and the shape of the deflected surface of the plate are related by a partial differential equation. An analysis of the problem involves finding a solution of the differential equation which also satisfies the specified edge conditions for the plate. After this deflection function is found, the bending moments may be determined from the second derivatives of the function. The derivation of the differential equation and the expressions for moments, shears, and reactions are given by various writers such as S. Timoshenko[2] and A. Nadai.[3]

INTRODUCTION

The theory of the bending of plates used in this paper is based upon several assumptions that must be considered in interpreting the results of the analysis. The principal assumptions may be noted briefly. The plate is assumed to be homogeneous, isotropic, and of constant thickness. The thickness of the plate is small compared with its lateral dimensions, so that the effect of shearing stresses on the deflection may be neglected. A line that is normal to the plane of the plate before bending remains straight and normal to the deflected middle surface after bending. The deflections are small compared with the thickness of the plate, so that direct stresses due to stretching of the middle surface may be neglected. A rough rule in this regard is that, if the deflection is less than half the plate thickness, the theory is satisfactory.[4] Most plates or slabs used in engineering structures conform to these assumptions closely enough so that analyses based on this theory serve as reliable guides for design. A. Nadai[3] and Nathan Newmark,[5] Assoc. M. Am. Soc. C. E., have presented detailed statements of the relation of the theory of plates to experiments and design.

In this particular investigation, the effect of Poisson's ratio has been neglected. No mathematical difficulties would be caused by including this item, but the arithmetical work would be greatly increased. Neglecting Poisson's ratio has no effect on the moments at the clamped edges but does affect, somewhat, the moments at interior points and the conditions at the free edge. Corrections to the interior moments for any given value of Poisson's ratio can be made.[2,3]

---

[2] "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1940.

[3] "Die elastischen Platten," by A. Nadai, Julius Springer, Berlin, 1925.

[4] "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, p. 51.

[5] "What Do We Know About Concrete Slabs?" by N. M. Newmark, *Civil Engineering*, September, 1940, pp. 559–562.

NOTATION

The letter symbols in this paper are defined where they first appear and are assembled for reference in the Appendix.

GENERAL PROCEDURE

This problem requires the solution of the differential equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

with the boundary conditions (assuming Poisson's ratio as zero):

At $x = 0.5\,a$,

$$w = 0 \qquad \text{and} \qquad \frac{\partial w}{\partial x} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2a)$$

At $y = b$,

$$w = 0 \qquad \text{and} \qquad \frac{\partial w}{\partial y} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2b)$$

At $y = 0$,

$$\frac{\partial^2 w}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^3 w}{\partial y^3} + 2\frac{\partial^3 w}{\partial x^2\,\partial y} = 0 \quad\ldots\ldots\ldots\ldots(2c)$$

In Eqs. 1 and 2: w is the deflection of the plate at any point; q is the intensity of a distributed lateral load; x and y are the rectangular coordinates in the plane of the middle surface of the plate; a and b are the dimensions of the plate in the x and y directions, respectively; and D, the flexural rigidity of the plate, is equal to

$$D = \frac{E\,h^3}{12\,(1 - \mu^2)} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

in which: E is the modulus of elasticity of the plate material; h is the thickness of the plate; and μ represents Poisson's ratio. The difficulty in solving this problem arises from the fact that there is no known procedure for solving the differential equation subject to these boundary conditions. The most general solution for rectangular plates is the one due to Levy[6] which uses for the deflection an expression of the form

$$w = \sum_{m=1}^{\infty} X_m \sin\frac{m\,\pi\,y}{b} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

in which X_m is a function of x alone; and m is an integer (which has the value 1, 2, 3 $\cdots$ to ∞ in successive terms of the series). The solution requires that the plate be simply supported along two edges and any boundary conditions may be prescribed at the other two edges. Therefore, it cannot be applied directly to the problem at hand. However, it is possible (in a manner somewhat analogous to that proposed by S. Timoshenko[7] for plates clamped at all four edges) to solve the problem by a proper combination of several loadings, each of which is of the form of Eq. 4.

---

[6] "Die elastischen Platten," by A. Nadai, Julius Springer, Berlin, 1925, p. 120.

[7] "Bending of Rectangular Plates with Clamped Edges," by S. Timoshenko, *Proceedings*, Fifth International Cong. for Applied Mechanics, Cambridge, Mass., 1938, p. 40.

For this purpose, the four cases shown in Fig. 2 are taken, each of which is simply supported along one pair of edges. Fig. 2(*a*) carries an hydrostatic loading, is simply supported at $x = \pm 0.5\,a$, fixed at $y = b$, and has a free edge at $y = 0$. Fig. 2(*b*) has no lateral load but is bent by arbitrary edge moments

$$(M_x)_{x=\pm\frac{a}{2}} = \sum_{m=1}^{\infty} F_m \sin\frac{m\,\pi\,y}{b} \qquad (5a)$$

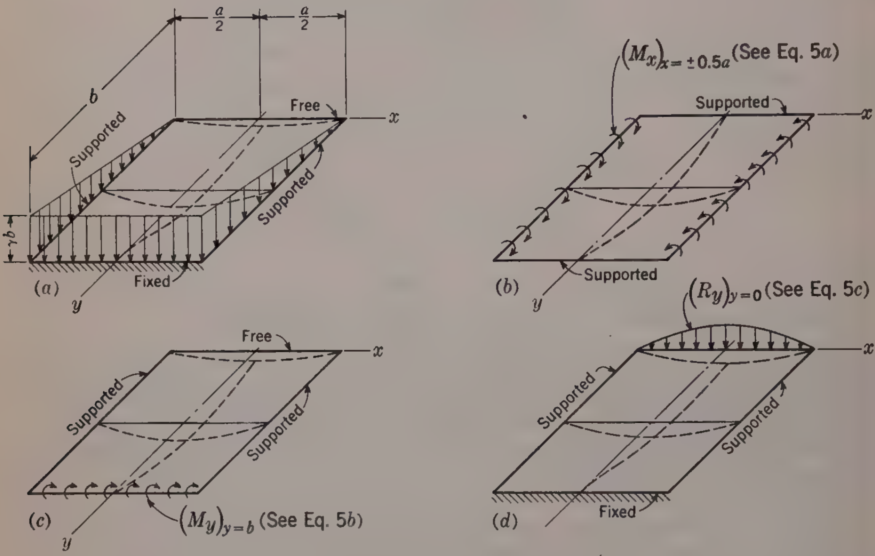


FIG. 2

the distribution of which along the edges is taken in the form of a trigonometric series with undetermined coefficients F_m. Fig. 2(*c*) is bent by edge moments

$$(M_y)_{y=b} = \sum_{k=1,3,5,\cdots} E_k \cos\frac{k\,\pi\,x}{a} \qquad (5b)$$

and Fig. 2(*d*) is bent by arbitrary edge reactions at $y = 0$ of the form

$$(R_y)_{y=0} = \sum_{k=1,3,5\cdots} R_k \cos\frac{k\,\pi\,x}{a} \qquad (5c)$$

In Eqs. 5, M_x and M_y are the bending moments per unit length of plate on sections perpendicular to the x and y axes, respectively; R_x and R_y are the reactions per unit length of plate along edges perpendicular to the x and y axes, respectively; m and k are integers, with values as indicated on the summation signs; and F_m, E_k, R_k, etc., are constants. The boundary conditions at the edges of these plates are as shown in Fig. 2. The deflection surface for each of these plates may be obtained by Levy's method and then these four solutions combined in such a manner that Eq. 1 and the boundary conditions, Eqs. 2, are satisfied. This leads to a set of simultaneous algebraic equations from which the values of the coefficients F_m, E_k, and R_k may be calculated.

The method of obtaining the deflection equation for the four cases will be illustrated by considering Fig. 2(*d*). Results for the other cases will then be given without proof since the same procedure applies.

DEFLECTION EQUATION FOR FIG. 2(*d*)

Denote the deflection in this case by w_4. Since there is no distributed lateral load on the plate, the differential equation (Eq. 1) reduces to

$$\frac{\partial^4 w_4}{\partial x^4} + 2\frac{\partial^4 w_4}{\partial x^2\, \partial y^2} + \frac{\partial^4 w_4}{\partial y^4} = 0 \quad \ldots\ldots (6)$$

The boundary conditions are:

At $x = \pm 0.5\, a$,

$$w_4 = 0 \quad \text{and} \quad \frac{\partial^2 w_4}{\partial x^2} = 0 \quad \ldots\ldots (7a)$$

At $y = b$,

$$w_4 = 0 \quad \text{and} \quad \frac{\partial w_4}{\partial y} = 0 \quad \ldots\ldots (7b)$$

At $y = 0$,

$$\frac{\partial^2 w_4}{\partial x^2} = 0 \quad \text{and} \quad -D\left(\frac{\partial^3 w_4}{\partial y^3} + 2\frac{\partial^3 w_4}{\partial x^2\, \partial y}\right) = \sum_{k=1,3,5\cdots} R_k \cos\frac{k\,\pi\,x}{a} \quad \ldots (7c)$$

Take the deflection function in the form

$$w_4 = \sum_{k=1,3,5\cdots} Y_k \cos\frac{k\,\pi\,x}{a} \quad \ldots\ldots (8)$$

in which Y_k is a function of y alone. Substituting Eq. 8 in Eq. 6, the function Y_k is found to be

$$\begin{aligned} Y_k = {} & G_k \cosh\frac{k\,\pi\,y}{a} + G'_k \frac{k\,\pi\,y}{a}\cosh\frac{k\,\pi\,y}{a} \\ & + G''_k \sinh\frac{k\,\pi\,y}{a} + G'''_k \frac{k\,\pi\,y}{a}\sinh\frac{k\,\pi\,y}{a} \quad \ldots\ldots (9) \end{aligned}$$

From the boundary conditions Eqs. 7*b* and 7*c*, the four constants in Eq. 9 are found to be

$$\left.\begin{aligned} G_k &= \frac{2\,a^3 R_k}{k^3 \pi^3 D\, K_k}\left[\tanh\alpha_k - \alpha_k\,(1 - \tanh^2\alpha_k)\right] \\ G'_k &= \frac{a^3 R_k}{k^3 \pi^3 D\, K_k}(2 - \tanh^2\alpha_k) \\ G''_k &= G'_k + \frac{a^3 R_k}{k^3 \pi^3 D} \\ G'''_k &= -0.5\, G_k \end{aligned}\right\} \quad \ldots\ldots (10)$$

in which

$$K_k = \tanh^2\alpha_k - \alpha^2_k\,(1 - \tanh^2\alpha_k) - 4 \quad \ldots\ldots (11)$$

α_k being equal to $\frac{k\,\pi\,b}{a}$.

Deflection Equation for Figs. 2(*a*), 2(*b*), and 2(*c*)

The deflection equations for the other cases are derived in a similar manner; thus, for Fig. 2(*a*):

$$w_1 = \frac{\gamma a^4}{\pi^5 D} \sum_{k=1,3,5\cdots} \frac{(-1)^{\frac{k-1}{2}}}{k^5} \left(4y + A_k \cosh \frac{k\pi y}{a} + A'_k \frac{k\pi y}{a} \cosh \frac{k\pi y}{a} + A''_k \sinh \frac{k\pi y}{a} + A'''_k \frac{k\pi y}{a} \sinh \frac{k\pi y}{a}\right) \cos \frac{k\pi x}{a} \qquad (12)$$

in which: $A_k = \frac{a\,\delta_k}{k\pi K_k}$; $A'_k = \frac{a\,\omega_k}{k\pi K_k}$; $A''_k = A'_k - \frac{8a}{k\pi}$; $A'''_k = -0.5\,A_k$;

and

$$\delta_k = 16\,\alpha_k (1 - \tanh^2 \alpha_k) - 16 \tanh \alpha_k + \frac{8}{\cosh \alpha_k}(\alpha_k - \tanh \alpha_k + \alpha^2_k \tanh \alpha_k) \qquad (13a)$$

$$\omega_k = 8 \tanh^2 \alpha_k - 16 + \frac{1}{\cosh \alpha_k}(8 + 4\,\alpha^2_k - 8\,\alpha_k \tanh \alpha_k) \qquad (13b)$$

and K_k is defined by Eq. 11.

Similarly, the deflection equation for Fig. 2(*b*) is:

$$w_2 = \frac{b^2}{2\pi^2 D} \sum_{m=1}^{\infty} \frac{F_m}{m^2 \cosh \beta_m} \times \left(\beta_m \tanh \beta_m \cosh \frac{m\pi x}{b} - \frac{m\pi x}{b} \sinh \frac{m\pi x}{b}\right) \sin \frac{m\pi y}{b} \qquad (14a)$$

and the deflection equation for Fig. 2(*c*) is:

$$w_3 = \frac{a^2}{\pi^2 D} \sum_{k=1,3,5\cdots} \frac{E_k}{k^2 \cosh \alpha_k} \left(B_k \cosh \frac{k\pi y}{a} + B'_k \frac{k\pi y}{a} \cosh \frac{k\pi y}{a} + B''_k \sinh \frac{k\pi y}{a} + B'''_k \frac{k\pi y}{a} \sinh \frac{k\pi y}{a}\right) \cos \frac{k\pi x}{a} \qquad (14b)$$

In Eqs. 14: $\beta_m = \frac{m\pi a}{2b}$; $\alpha_k = \frac{k\pi b}{a}$; and—

$$\left.\begin{aligned} B_k &= \frac{\alpha_k + \tanh \alpha_k}{3 \tanh \alpha_k + \alpha_k (1 - \tanh^2 \alpha_k)} \\ B'_k &= \frac{0.5\,\alpha_k \tanh \alpha_k - 1}{3 \tanh \alpha_k + \alpha_k (1 - \tanh^2 \alpha_k)} \\ B''_k &= B'_k \\ B'''_k &= -0.5\,B_k \end{aligned}\right\} \qquad (15)$$

EQUATIONS FOR DETERMINING F_m, E_k, AND R_k

The solution for the given problems is to be obtained by superposition of the solutions for the four cases in Fig. 2, by taking

$$w = w_1 + w_2 + w_3 + w_4 \qquad (16)$$

Coefficients F_m, E_k, and R_k are to be so determined that the boundary conditions in Eq. 2 will be satisfied. Thus, for the condition that the slope is zero at $y = b$,

$$\left(\frac{\partial w_2}{\partial y}\right)_{y=b} + \left(\frac{\partial w_3}{\partial y}\right)_{y=b} = 0 \qquad (17)$$

Differentiating, Eq. 14*a* gives

$$\left(\frac{\partial w_2}{\partial y}\right)_{y=b} = \frac{b}{2\,\pi\,D} \sum_{m=1}^{\infty} \frac{(-1)^m F_m}{m \cosh \beta_m}$$
$$\times \left(\beta_m \tanh \beta_m \cosh \frac{m\,\pi\,x}{b} - \frac{m\,\pi\,x}{b} \sinh \frac{m\,\pi\,x}{b}\right) \qquad (18a)$$

and from Eq. 14*b*

$$\left(\frac{\partial w_3}{\partial y}\right)_{y=b} = \frac{a}{2\,\pi\,D} \sum_{k=1,3,5\cdots} \frac{E_k}{k} \frac{(1 + \alpha^2_k)(\tanh^2 \alpha_k - 1) - 3}{3 \tanh \alpha_k + \alpha_k (1 - \tanh^2 \alpha_k)} \cos \frac{k\,\pi\,x}{a} \qquad (18b)$$

Before substituting the two slope expressions in Eq. 17, it is expedient to expand the expression inside the brackets in Eq. 18*a* in a Fourier cosine series. Doing this, and then substituting in Eq. 17 the following equation is obtained connecting the coefficients E_k and F_m:

$$E_k \frac{(1 + \alpha^2_k)(1 - \tanh^2 \alpha_k) + 3}{3 \tanh \alpha_k + \alpha_k (1 - \tanh^2 \alpha_k)} = \frac{8\,(-1)^{\frac{k-1}{2}}\,k^2\,b^3}{\pi\,a^3} \sum_{m=1}^{\infty} \frac{(-1)^m\,m\,F_m}{\left(m^2 + \frac{k^2\,b^2}{a^2}\right)^2} \qquad (19)$$

There will be one equation of the form of Eq. 19 for each value of $k = 1, 3, 5, \cdots$, and each of these will have an infinite number of terms on the right-hand side.

In a similar manner, from the condition that the reaction

$$R_y = -D\left(\frac{\partial^3 w}{\partial y^3} + 2\,\frac{\partial^3 w}{\partial x^2\,\partial y}\right) \qquad (20a)$$

at $y = 0$ must be zero, the following system of equations is obtained:

$$R_k = -\frac{4\,(-1)^{\frac{k-1}{2}}\,k\,b}{a^2} \sum_{m=1}^{\infty} m\,F_m \frac{m^2 + \frac{2\,k^2\,b^2}{a^2}}{\left(m^2 + \frac{k^2\,b^2}{a^2}\right)^2} \qquad (20b)$$

From the condition that the slope at $x = 0.5\,a$ must be zero:

$$\begin{aligned}
&F_m \left(\tanh \beta_m + \frac{\beta_m}{\cosh^2 \beta_m} \right) + \frac{4\,m^2\,a^3}{\pi\,b^3} \sum_{k=1,3,5\cdots} \frac{k\,(-1)^{\frac{k-1}{2}}\,E_k}{\left(k^2 + \frac{m^2\,a^2}{b^2} \right)^2} \\
&\quad \times \left\{ \frac{(\alpha_k + \tanh \alpha_k) \left(2 + \frac{m^2\,a^2}{k^2\,b^2} \right)}{[3 \tanh \alpha_k + \alpha_k\,(1 - \tanh^2 \alpha_k)] \cosh \alpha_k} - (-1)^m \right\} \\
&+ \frac{8\,m^2\,a^4}{\pi^2\,b^3} \sum_{k=1,3,5\cdots} \frac{(-1)^{\frac{k-1}{2}}\,R_k}{K_k \left(k^2 + \frac{m^2\,a^2}{b^2} \right)^2} \left\{ [\tanh \alpha_k - \alpha_k\,(1 - \tanh^2 \alpha_k)] \right. \\
&\quad \left. \times \left(2 + \frac{m^2\,a^2}{k^2\,b^2} \right) + (-1)^m \frac{\alpha_k + \tanh \alpha_k}{\cosh \alpha_k} \right\} = - \frac{4\,\gamma\,a^4}{\pi^4\,b} \sum_{k=1,3,5\cdots} \frac{1}{k^3} \\
&\quad \times \left\{ - \frac{(-1)^m\,4\,k\,\frac{b}{a}}{k^2 + \frac{m^2\,a^2}{b^2}} + \frac{\frac{m^2\,a^2}{\pi\,b^2}}{\left(k^2 + \frac{m^2\,a^2}{b^2} \right)^2 K_k} \left[\delta_k \left(2 + \frac{m^2\,a^2}{k^2\,b^2} \right) \right. \right. \\
&\qquad \left. \left. + (-1)^m \cosh \alpha_k\,(-\delta_k + 2\,\omega_k \tanh \alpha_k) \right] \right\} \qquad \text{(21)}
\end{aligned}$$

Eqs. 19, 20*b*, and 21 provide the necessary relations for determining F_m, E_k, and R_k. Each one defines an infinite system of equations with an infinite number of terms. To solve these numerically, it is necessary to assume that the series converge and to neglect all terms higher than a given order. The result is a finite system of simultaneous equations that may be solved to obtain values for the coefficients. This completes the solution of the differential Eq. 1 for the specified boundary conditions. The moments at any point may then be determined from the known relations of the plate theory. For example, the moment per unit length of plate on sections perpendicular to the x-axis is given by

$$M_x = - D \frac{\partial^2 w}{\partial x^2} = - D \left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_3}{\partial x^2} + \frac{\partial^2 w_4}{\partial x^2} \right) \qquad \text{(22)}$$

assuming that Poisson's ratio is zero.

NUMERICAL EXAMPLE

For an example, assume a tank with square sides for which $a = b$; and in each of Eqs. 19, 20*b*, and 21 take only the first four terms of each series. Then in Eq. 19, using $m = 1, 2, 3$, and 4, and taking $k = 1, 3, 5$, and 7, successively, the numerical values of the constants may be computed to obtain the four equations:

$$\begin{array}{rl}
1.0228\,E_1 = & -\,0.6366\,F_1 + 0.2037\,F_2 - 0.0764\,F_3 + 0.0352\,F_4 \\
1.0000\,E_3 = & 0.2292\,F_1 - 0.2712\,F_2 + 0.2122\,F_3 - 0.1467\,F_4 \\
1.0000\,E_5 = & -\,0.0942\,F_1 + 0.1514\,F_2 - 0.1652\,F_3 + 0.1515\,F_4 \\
1.0000\,E_7 = & 0.0499\,F_1 - 0.0888\,F_2 + 0.1113\,F_3 - 0.1181\,F_4
\end{array} \qquad \text{(23a)}$$

Calculation of the constants in Eq. 20*b* gives the system:

$$\left.\begin{array}{l} a\,R_1 = -\,3.0000\,F_1 - 1.9200\,F_2 - 1.3200\,F_3 - 0.9965\,F_4 \\ a\,R_3 = 2.2800\,F_1 + 3.1243\,F_2 + 3.0000\,F_3 + 2.6112\,F_4 \\ a\,R_5 = -\,1.5089\,F_1 - 2.5683\,F_2 - 3.0623\,F_3 - 3.1410\,F_4 \\ a\,R_7 = 1.1088\,F_1 + 2.0335\,F_2 + 2.6718\,F_3 + 3.0220\,F_4 \end{array}\right\}\ldots(23b)$$

Finally, from Eq. 21, compute the following set of numerical equations:

$$\left.\begin{array}{l} 1.1666\,F_1 + 0.4314\,E_1 - 0.0382\,E_3 + 0.0094\,E_5 \\ \qquad - 0.0036\,E_7 - 0.1685\,a\,R_1 + 0.0057\,a\,R_3 \\ \qquad\qquad - 0.0008\,a\,R_5 + 0.0002\,a\,R_7 = -\,0.04694\,\gamma\,a^3 \\ 1.0197\,F_2 - 0.0589\,E_1 + 0.0903\,E_3 - 0.0303\,E_5 \\ \qquad + 0.0127\,E_7 - 0.2608\,a\,R_1 + 0.0156\,a\,R_3 \\ \qquad\qquad - 0.0028\,a\,R_5 + 0.0008\,a\,R_7 = -\,0.03474\,\gamma\,a^3 \\ 1.0014\,F_3 + 0.2640\,E_1 - 0.1063\,E_3 + 0.0496\,E_5 \\ \qquad - 0.0238\,E_7 - 0.2450\,a\,R_1 + 0.0225\,a\,R_3 \\ \qquad\qquad - 0.0050\,a\,R_5 + 0.0016\,a\,R_7 = -\,0.02586\,\gamma\,a^3 \\ 1.0001\,F_4 + 0.0799\,E_1 + 0.0976\,E_3 - 0.0606\,E_5 \\ \qquad + 0.0338\,E_7 - 0.2603\,a\,R_1 + 0.0261\,a\,R_3 \\ \qquad\qquad - 0.0068\,a\,R_5 + 0.0024\,a\,R_7 = -\,0.03485\,\gamma\,a^3 \end{array}\right\}\ldots(23c)$$

There are now 12 unknowns in 12 simultaneous equations. Solving these equations yields the following values for the coefficients:

$$\begin{array}{lll} F_1 = -\,0.0314\,\gamma\,a^3 & E_1 = 0.0190\,\gamma\,a^3 & R_1 = 0.1076\,\gamma\,a^2 \\ F_2 = -\,0.0030\,\gamma\,a^3 & E_3 = -\,0.0063\,\gamma\,a^3 & R_3 = -\,0.0999\,\gamma\,a^2 \\ F_3 = -\,0.0026\,\gamma\,a^3 & E_5 = 0.0023\,\gamma\,a^3 & R_5 = 0.0765\,\gamma\,a^2 \\ F_4 = -\,0.0043\,\gamma\,a^3 & E_7 = -\,0.0011\,\gamma\,a^3 & R_7 = -\,0.0608\,\gamma\,a^2 \end{array}$$

The bending moments and the deflection at any point may now be computed, the results being shown in Figs. 3 and 4.

The numerical values given are approximate due to the fact that only four terms were taken in each series. However, each of the functions used satisfies the differential equation and gives zero deflection along the sides and bottom of the plate. The inaccuracy arises from the fact that the slope at these edges will not quite be zero, and the top edge will not quite be free, unless an infinite number of terms is used. An insight into the error involved may be

TABLE 1.—NUMERICAL VALUES[a] FOR EDGE SLOPES

LOCATION		EDGE SLOPES FOR PLATES SHOWN IN:				Resultant slope
x	y	Fig. 2(*a*)	Fig. 2(*b*)	Fig. 2(*c*)	Fig. 2(*d*)	
0	*b*	0	0.00281	−0.00280	0	0.00001
0.5 *a*	0	−0.00783	0	−0.00072	0.00797	−0.00058
0.5 *a*	0.25 *b*	−0.00773	0.00446	−0.00089	0.00428	0.00012
0.5 *a*	0.50 *b*	−0.00663	0.00569	−0.00113	0.00202	0.00005
0.5 *a*	0.75 *b*	−0.00336	0.00401	−0.00122	0.00060	0.00003

[a] Numerical values in this table are coefficients that should be multiplied by $\frac{\gamma\,a^4}{D}$ to obtain slope.

obtained by computing the actual slope along the edges for the problem as solved. Results of such a calculation are given in Table 1. From this it is seen that the actual edge slopes remaining are only a few per cent of the corresponding slopes for Fig. 2(*a*) which has simply supported edges. So it may be concluded that the values found in this example are sufficiently accurate for design purposes.

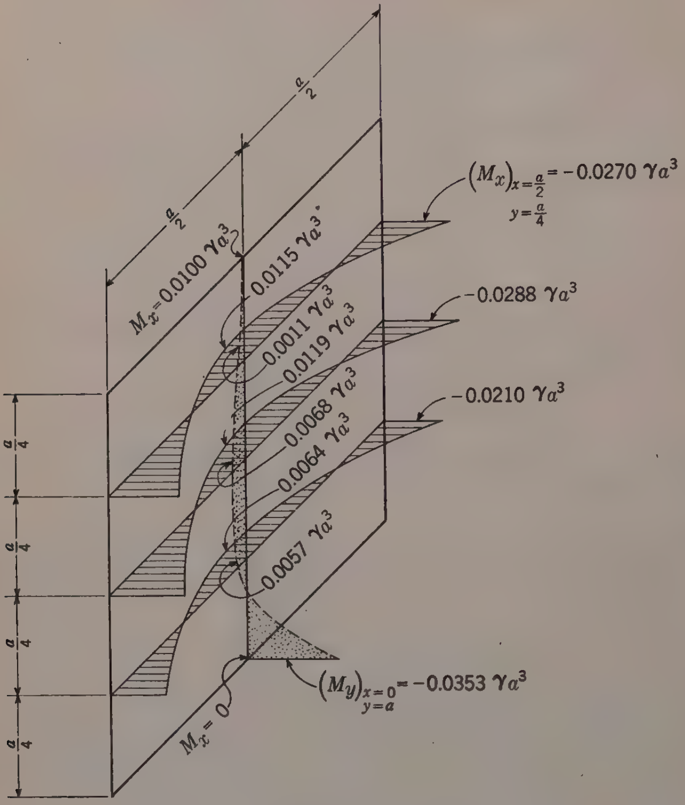


FIG. 3

To repeat: The ordinary theory of bending of plates, and the results found from it, are valid only if the deflections are small compared to the wall thick-

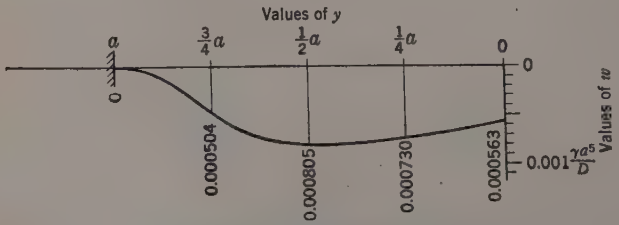


FIG. 4.—DEFLECTION ALONG VERTICAL CENTER LINE

ness. This condition may be readily investigated for any particular design. For example, take a steel water tank with sides 100 in. square. The maximum moment per linear inch is $M = -0.0353\,\gamma\,a^3 = -0.0353 \times 100^3 \times \frac{62.5}{1{,}728}$ $= -1{,}280$ in-lb per in. Assuming an allowable bending stress of 18,000 lb per sq in., $\frac{18{,}000\,h^2}{6} = 1{,}280$; from which $h = 0.654$ in., say $\frac{11}{16}$ in. The maximum deflection is given by $w = 0.000805\,\frac{\gamma\,a^5}{D} = 0.33$ in. using $E = 30{,}000{,}000$ lb per sq in. and $\mu = 0.3$. This gives a ratio of deflection to wall thickness of $\frac{0.33}{0.6875} = 0.48$ which is just about the maximum permissible.

OTHER SOLUTIONS

This same problem has been solved by other investigators using different methods. P. T. Cheng[8] approached the problem by choosing deflection functions containing arbitrary constants which he determined numerically by satisfying the boundary conditions at a finite number of points along the edges. His final values for the moments and deflections check, very closely, the values given herein. Calculations by the method of finite differences have been made by A. Smotrow[9] for the same problem, using a network of 36 squares. The moment values thus obtained are appreciably smaller than those found in this paper. For example, the maximum moment at the base is given by Smotrow as $0.030\,\gamma\,a^3$ as compared to $0.0353\,\gamma\,a^3$ given in Fig. 3. This is to be expected, and it is believed that a closer network of points would have to be taken in the finite difference method to obtain the accuracy of the superposition method used herein. Smotrow also completed the calculations for plates with $\frac{b}{a}$-ratios of 1.5 and 0.67.

MOMENTS FOR PLATES OF OTHER PROPORTIONS

Calculations have been given, in this paper, only for a square plate in which the ratio $\frac{b}{a}$, of the height to the width, is unity. Having the moment values for this case as a standard of comparison, it is possible to approximate moments for plates with other $\frac{b}{a}$-ratios by the consideration of known solutions of plates with the same loading but having other edge conditions.

Consider first the moment M_y at the center of the base of Figs. 5(*a*), 5(*b*), and 5(*c*), all of which carry the same hydrostatic load. Fig. 5(*a*) is the one treated in this paper. Fig. 5(*b*) is similar except that the top edge is simply supported, whereas Fig. 5(*c*) has simply supported side edges and is the same as Fig. 2(*a*). It is clear that the base moment for Fig. 5(*a*) will be greater than for Fig. 5(*b*) and less than for Fig. 5(*c*). For values of $\frac{b}{a}$ greater than unity, the moment for Fig. 5(a) will approach that of Fig. 5(b), and for values of $\frac{b}{a}$ less than unity, it will approach that of Fig. 5(c). Moments for Fig. 5(b) with various $\frac{b}{a}$-ratios have been published,[10] and values for Fig. 5(c) may be readily calculated from Eq. 12. In Fig. 6 estimated values for Fig. 5(a) are indicated by the broken curve drawn through the control point at $\frac{b}{a} = 1$ and approaching the other two curves for other values of $\frac{b}{a}$.

---

[8] "The Analysis of Rectangular Plates Under Various Loading and Boundary Conditions," by P. T. Cheng, unpublished research work at Univ. of Minnesota, Minneapolis, Minn., 1940.

[9] "Berechnung von Platten," by A. Smotrow (in Russian), *Staats-Verlag für Literatur des Bauwesens*, Moskau und Leningrad, 1936.

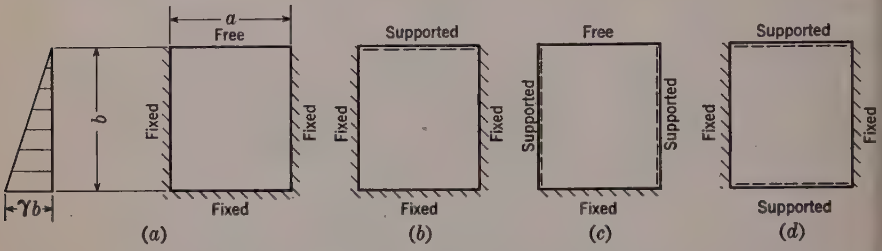


FIG. 5

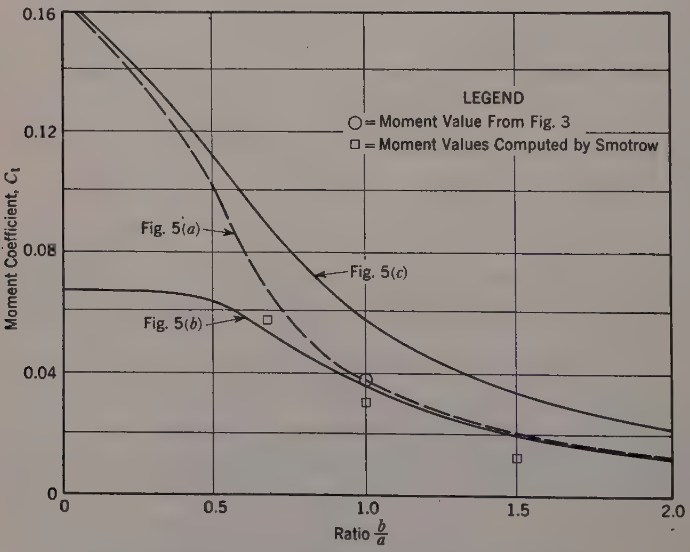


FIG. 6.—MOMENT AT CENTER OF BASE: $M_y = -C_1 \gamma b^3$

For the edge moment at the midpoint of each side, it seems reasonable to assume that the value for Fig. 5(a) will lie between that for Fig. 5(b) and that for Fig. 5(d). Values for the case represented by Fig. 5(d) are given by Professor Timoshenko.[11] Curves for Figs. 5(*b*) and 5(*d*) are plotted in Fig. 7 from data in the references cited and the assumed curve for Fig. 5(*a*) is indicated by the broken line.

---

[10] "Analysis of Clamped Rectangular Plates," by D. Young, *Journal of Applied Mechanics*, December, 1940, p. A-139.

It is believed that these moment values are adequate for design purposes. For plates with $\frac{b}{a}$-ratios not too different from unity $\left(\text{say from } \frac{2}{3} \text{ to } \frac{3}{2}\right)$ the distribution of moments along the base may be taken as parabolic and along the sides may be conservatively considered as constant over the middle half and then decreasing uniformly to one half its maximum value at either end.

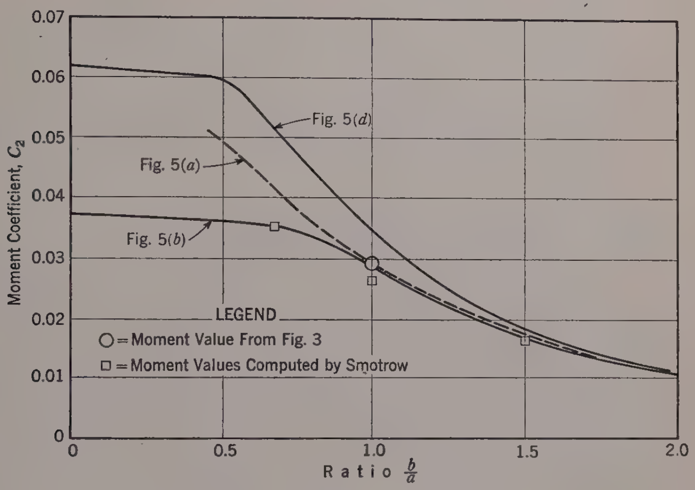


Fig. 7.—Moment at Center of Side: $M_x = -C_2 \gamma b^3$

APPENDIX

Notation

The following letter symbols, adopted for use in this paper, conform essentially to Standard Letter Symbols for Structural Analysis, prepared by a Committee of the American Standards Association with Society representation and approved by the Association in 1941.[12] A few obvious conflicts—such as D (standard symbol for diameter) = flexural rigidity and w (standard symbol for uniformly distributed load) = deflection of plate—have been introduced to adapt the notation to common terms found in current literature on flat plate analysis.

A = a constant; with subscripts m or k, the symbols A, B, E, F, G, K, R, δ, and ω are each constants defined where they are used;

---

[11] "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., p. 210.

[12] Publication pending.

a = length of the plate in the x-direction;
B = (see A);
b = length of the plate in the y-direction;
D = flexural rigidity of the plate (see Eq. 3);
E = modulus of elasticity of the plate material; see also under A;
F = (see A);
G = (see A);
h = thickness of the plate;
K = (see A);
k = an integer; see also under A;
M = bending moment; M_x and M_y are bending moments per unit length of plate on sections perpendicular to the x-axis and y-axis, respectively;
m = an integer; see also under A;
q = intensity of a distributed lateral load;
R = unit reactions; R_x and R_y are the reactions per unit length of the plate along edges perpendicular to the x-axis and y-axis, respectively; see also under A;
w = deflection of the plate at any point;
X = a function of x;
x = an abscissa; x and y = rectangular coordinates in the plane of the middle surface of the plate;
Y = a function of y;
y = (see x);
$\alpha_k = \frac{k \pi b}{a}$;
$\beta_m = \frac{m \pi a}{2 b}$;
γ = specific weight of water; weight per unit volume of water;
δ = (see A);
μ = Poisson's ratio;
ω = (see A).

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
PAPERS

ENERGY LOSS AT THE BASE OF A FREE OVERFALL

BY WALTER L. MOORE,[1] JUN. AM. SOC. C. E.

SYNOPSIS

Experimental studies were made of a free overfall with a view to obtaining information that would be of value to designers of hydraulic structures. Detailed laboratory measurements showed that the energy losses at the base of a fall were of appreciable magnitude and hence must be considered in hydraulic design. These measured energy losses were applied in the development of a rational formula for calculating the height of the jump below a fall. Limited information was also obtained on the length characteristics of the jump and on the effect of submergence of the jump on energy dissipation. The presence of standing water behind the fall is explained, and its height is calculated by application of the momentum equation.

INTRODUCTION

Hydraulic model studies have been used with much success in the design of spillway structures and outlet works of all sizes and types. Ordinarily, they are used only after the more common and much cheaper analytical design methods, which are based on experience, research, and analysis, have been investigated and considered inadequate. The extension of the common design methods to cover problems which at present can be solved satisfactorily only by the use of models will come, along with an appreciable saving of time and expense, when the proper fundamental information is made available. The use of the analytical methods that have a sound basis not only saves the cost of the model studies but, in general, results in more reliable designs.

The experimental study reported herein was made at the Cooperative Laboratory of the Soil Conservation Service, U. S. Department of Agriculture, and the California Institute of Technology as part of a general program, the objective of which is to supply the engineer with basic information that can be applied in the design of hydraulic structures. Since the overfall is used

---

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **March, 1942.**

[1] Junior Engr., SCS Laboratory, California Inst. of Technology, Pasadena, Calif.

frequently in soil conservation work, it was selected for study. Stress has been laid upon energy loss or energy dissipation not only because it is of importance in itself, but also because it can be used in calculating the flow characteristics, which in turn determine the design. The results are of special interest to technicians concerned with the design of small dams used in gully control work, as well as to engineers dealing with other hydraulic structures where the overfall is used to dissipate energy.

This investigation was conducted by the usual experimental method, which combines physical measurements with analysis. It was decided to study first a structure that had been simplified as much as possible in the conviction that the simple structure must be thoroughly understood before a more complicated one can be studied intelligently. Following this method the complications introduced by side contractions and expansions were eliminated in an effort to approach the condition of two-dimensional flow. Likewise, the effects of sills and pools were not included in the study.

APPARATUS AND PROCEDURE

A free overfall model was constructed having a rectangular channel 11 in. wide, discharging into a similar channel at a lower elevation. The channel sides were carefully joined to form two smooth parallel planes with which the edges of the falling jet were in continuous contact. One side of the model was of plate glass and permitted direct observation of the flow pattern (see Fig. 1).

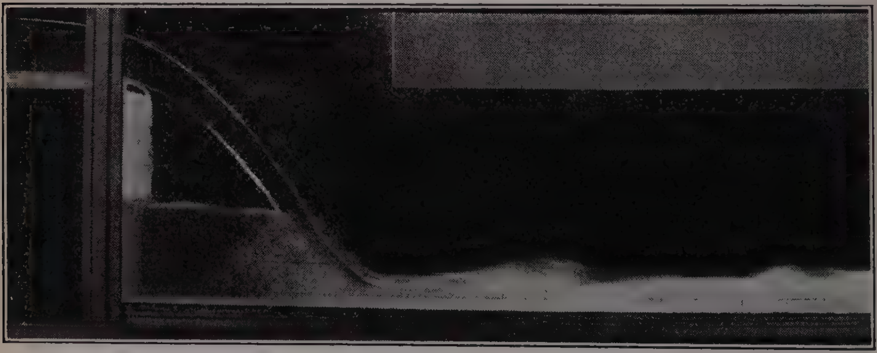
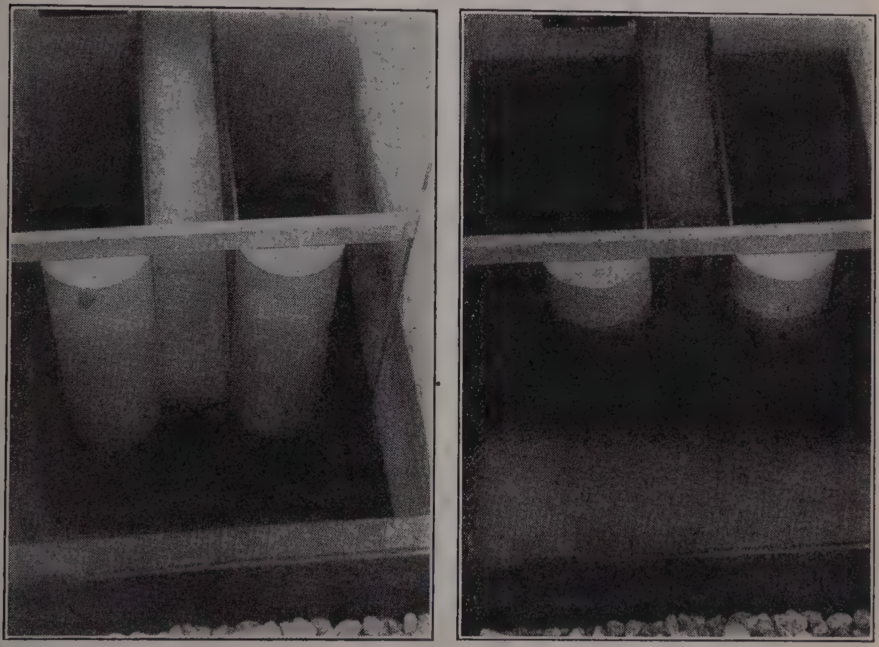


FIG. 1.—FREE OVERFALL MODEL; VERTICAL STRINGS ARE SPACED AT 0.5 FT

The lower channel could be raised and lowered to vary the height of fall to a maximum of 1.5 ft. Water was supplied by an electrically driven centrifugal pump discharging into a constant head tank from which the flow was conducted through an 8-in. pipe to the forebay of the model. The discharge was regulated with an 8-in. gate valve, and measured by a venturi meter installed in the supply line. The differential head on the venturi meter could be read to 0.001 ft on either a water or mercury-filled manometer, which made it possible to set the rate of flow to within 1% from 0.15 cu ft per sec to 1.85 cu ft per sec. The disturbance caused as the 8-in. pipe discharged into the forebay was stilled very effectively by passing the flow through a 1-ft wall of loosely piled $1\frac{1}{2}$-in. rock. From the tranquil forebay a transition section led the flow into the model channel with a minimum of disturbance (see Fig. 2). Elevations were determined to 0.001 ft with a point gage mounted upon an instrument carriage that could be rolled along rails above the channel. On the center line of the lower channel a brass strip with forty piezometer openings was installed in order to investigate the longitudinal variation of the bottom pressure. The depth of tailwater in the downstream channel was controlled by the wicket-type tail gate shown in Fig. 3. This gate, which caused minimum disturbance to the flow pattern upstream, was located 11 ft from the fall and 3 ft downstream from the last section at which measurements were taken.



(*a*) (*b*)

FIG. 2.—VIEW OF MODEL FACING DOWNSTREAM ON FOREBAY: (*a*) TRANSITION SECTION IN EMPTY FOREBAY; (*b*) UNDISTURBED FLOW ENTERING TRANSITION

When the depth of flow would permit, velocities were measured with a standard Prandtl pitot-static tube that had a coefficient of 1.00 (Fig. 4(*a*)). In order to determine velocities in shallow flows, a hypodermic needle was used to measure the total head (Fig. 4(*b*)) and a piezometer was used to obtain the static head. By using hypodermic needles of three sizes (16, 18, and 19, Birmingham wire gage) the diameter of the measuring tube was kept within one tenth of the depth of flow. In regions where the bottom piezometer could not be relied upon for the static pressure, measurements were made with a hypodermic needle closed at the end and drilled with static holes according to the standards given by H. Peters[2] (Fig. 4(C)). This apparatus was well adapted to a detailed investigation of a two-dimensional drop and allowed particular attention to be given to the energy losses involved.

---

[2] "Druckmessung," von H. Peters, *Handbuch der Experimentalphysik*, IV, I Teil, pp. 489–509.

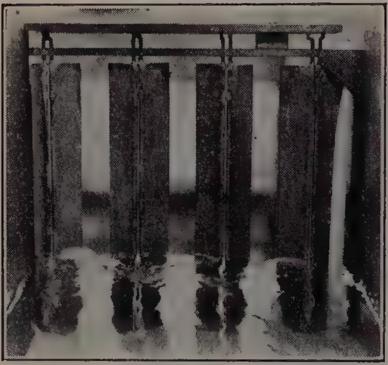


FIG. 3.—UPSTREAM FACE OF TAIL GATE WITH WICKETS PARTLY CLOSED TO RAISE TAILWATER

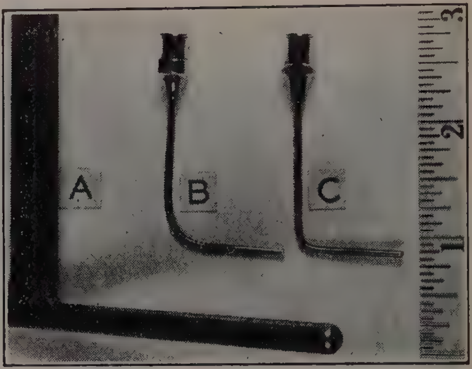


FIG. 4.—TUBES USED IN MAKING VELOCITY MEASUREMENTS: (A) PRANDTL PITOT-STATIC TUBE; (B) AND (C) HYPODERMIC NEEDLE FOR READING TOTAL AND STATIC HEAD, RESPECTIVELY

NOTATION

The letter symbols in this paper are defined where they first appear and are assembled for reference in the Appendix.

PREVIOUS STUDY

Very little study has been given to the problem of a free overfall striking a flat bottom. Boris A. Bakhmeteff,[3] M. Am. Soc. C. E., has shown that the energy equation may be applied to determine the flow conditions at the toe of a fall. Fig. 5 is a sketch of the water profile for a typical overfall with the downstream depth, d_1, unaffected by the tailwater in the channel—that is, with the flow in the downstream channel at supercritical velocity.

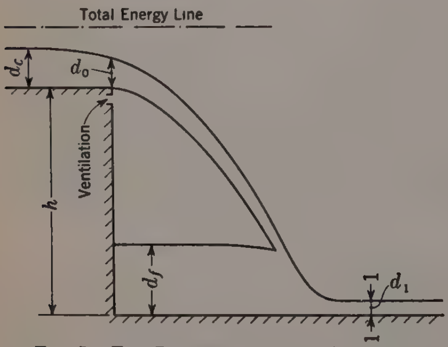


FIG. 5.—TWO-DIMENSIONAL FREE OVERFALL

Assuming hydrostatic pressure and uniform velocity distribution, the energy head E_0 at the upper level is

$$E_0 = h + \frac{3}{2} d_c \quad \text{(1)}$$

in which d_c is the critical depth and h is the height of the fall. The velocity at section 1–1 may be expressed as

$$V_1 = \phi \sqrt{2 g (E_0 - d_1)} \quad \text{(2)}$$

in which ϕ is a velocity coefficient, g is acceleration due to gravity, and d_1 is the depth at section 1–1. Transposing and introducing the expression for E_0 into Eq. 2, there follows

---

[3] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., 1932, p. 295.

$$\frac{V^2_1}{2\,g} = \phi^2 \left(h + \frac{3}{2}\,d_c - d_1 \right) \qquad (3)$$

Introducing the continuity relation, $V_1\,d_1 = d_c\,\sqrt{g\,d_c}$, and rearranging Eq. 3 give

$$\left(\frac{d_c}{d_1}\right)^3 - \phi^2\,\frac{d_c}{d_1}\left(\frac{2\,h}{d_c} + 3\right) + 2\,\phi^2 = 0 \qquad (4)$$

Eq. 4 may be transformed to agree identically with the equation given by Professor Bakhmeteff. It is presented herein in a dimensionless form in which the ratio $\frac{d_c}{d_1}$ describes the flow conditions at the base of the fall, and $\frac{h}{d_c}$ represents the relative height of fall. Professor Bakhmeteff states that the coefficient ϕ may be taken equal to unity, which means there is no loss up to section 1–1, Fig. 5. P. M. Bobin,[4] who uses a similar method of attack, makes no statement as to the correct value of ϕ, but in one example uses a value of $\phi = 0.95$. So far as the writer knows, an experimental determination of the energy loss in a free overfall has not been made previously. Since the flow characteristics in the remainder of the structure are dependent upon the energy loss at the base of the fall, the first aim of this investigation was to determine this loss and thus make it possible to calculate the vertical elements of a simple two-dimensional drop structure.

FLOW CONDITIONS WITH SHOOTING TAILWATER

When there was shooting flow in the lower channel, the energy loss at the base of the fall could be most easily investigated because the area being studied was unaffected by downstream disturbances.

Energy Loss at Base of Fall.—When these studies were begun, it was felt that the energy content in the lower channel could be determined by obtaining the value of d_1 and applying the relation

$$E_1 = d_1 + \frac{V^2_1}{2\,g} = d_1 + \frac{q^2}{2\,g\,d^2} = d_1 + \frac{d^3_c}{2\,d^2_1} \qquad (5)$$

in which q is discharge per foot of width. It should be noted that E_1 is very sensitive to changes in the value of d_1. Obviously, then, the accurate measurement of d_1 was essential, but the presence of heavy spray near the base of the fall prevented this by introducing errors which, in the case of thin flows and high falls, exceeded 25%. Furthermore, Eq. 5 is based upon the assumption of uniform velocity. When an investigation revealed that the velocity profile below the fall was not uniform and, consequently, that the basic assumption was not valid under the existing circumstances, this approach to the problem was abandoned.

It was found necessary to evaluate the velocity energy directly by taking velocity measurements in the thin sheet of water shooting away from the toe of the fall. This was accomplished by using a hypodermic needle to obtain the total head and bottom piezometers to determine the static head. For points not located on the center line over the piezometers, an adjustment based on the transverse variation of depth was made in the static head reading. For each of three stations in the downstream channel, velocity profiles were taken at the lateral odd tenth-points, and plots were made showing values of velocity, V, and velocity cubed, V^3. The velocity energy at each station was computed from the summation, $\frac{V^3\,\Delta A}{2\,g\,V\,\Delta A}$, which gives the average kinetic energy per pound of fluid flowing across the section. Measurements also were taken of the surface elevation as given by the point gage, and of the bottom pressure as given by the piezometers. The data for each run were plotted as shown in Fig. 6, in which Z = the elevation of the plane of the horizontal velocity traverse; h_v = venturi meter reading = 0.42 in. of mercury; Q = 1.218 cu ft per sec; d_c = 0.380 ft; d_0 = 0.272 ft; d = 0.152 ft at 5.0 ft from the fall; and $\frac{\sum(V^3\,\Delta A)}{2\,g\sum(V\,\Delta A)} + d$ = specific energy. Six runs were made with the fall height of 1.5 ft covering a range of $\frac{h}{d_c}$ from 3.0 to 14.6. The value of d_c varied from 0.50 ft to 0.103 ft. The fall height was then decreased to 0.5 ft, and five runs were made that covered a range of ratios $\frac{h}{d_c}$ from 1.0 to 4.85. (A minimum value of 0.1 ft was set for d_c in order to avoid the range where viscosity and

---

[4] "The Design of Stilling Basins," by P. M. Bobin, *Transactions*, Scientific Research Inst. of Hydrotechnics, Vol. XIII, 1934, Leningrad, pp. 79–123.

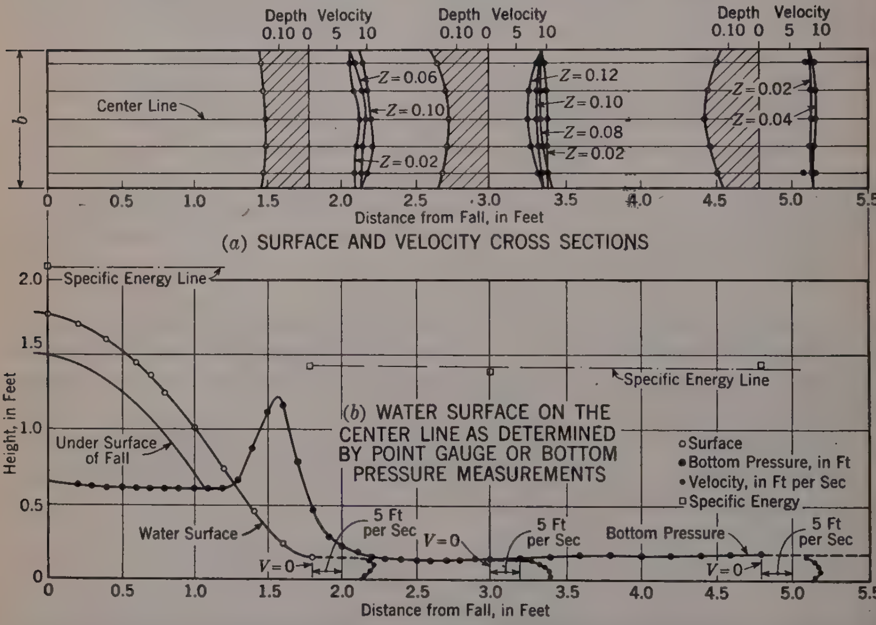


Fig. 6.—Typical Plot of Surface Profile, Pressure, and Velocity Measurements Through Fall

surface tension exert appreciable effects on the flow. Because extremely small depths were avoided, and because of the disturbed nature of the flow below the fall, negligible "scale effects" are to be expected, and the results can be applied to actual field cases without modification.) The energy E_1 at the base of the fall was taken as the sum of the integrated velocity energy and the average depth of flow at the section. The experimental values of $\frac{E_1}{d_c}$ are plotted against $\frac{h}{d_c}$ in curve (2), Fig. 7(*a*). In order to compare this curve with the values obtained from the energy equation, assuming no loss, curve (1), Fig. 7(*a*), was plotted from the equation:

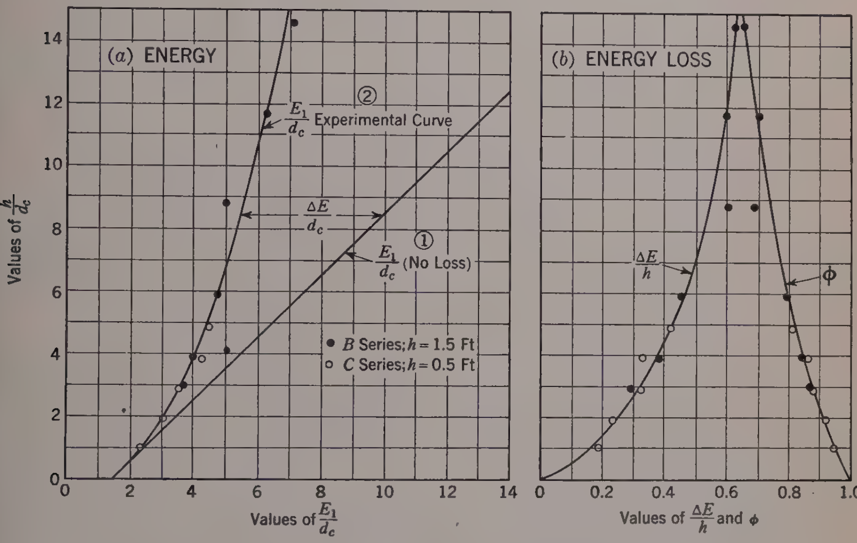


FIG. 7.—ENERGY AT THE BASE OF THE FALL

$$E_1 = h + \frac{3}{2} d_c \quad \text{(6a)}$$

or

$$\frac{E_1}{d_c} = \frac{h}{d_c} + \frac{3}{2} \quad \text{(6b)}$$

The difference between these two curves represents the energy loss divided by the critical depth, $\frac{\Delta E}{d_c}$. To show more clearly the effect of $\frac{h}{d_c}$ on the rate of energy loss, Fig. 7(*b*) was prepared, in which the energy loss is presented as a fraction of the fall height. Although this indicates that as much as two thirds of the energy released by the fall may be dissipated as the jet strikes the flat bottom, for a small value of $\frac{h}{d_c}$ (say 1.5) only one fifth of the energy is dissipated. With a given discharge and fall height, more effective dissipation may be achieved by increasing the width of the overfall section, and thereby reducing the value of d_c.

Values of the velocity coefficient ϕ were calculated from the observed energy loss and plotted in Fig. 7(*b*) as a function of $\frac{h}{d_c}$. Since the use of ϕ in Eq. 4 involves solving a cubic equation, this curve is of less practical value than curve (2), Fig. 7(*a*), which gives, directly, the energy content at the base of the fall.

Standing Water Behind Fall.—In all runs in this series it was noted that the depth of water behind the fall was considerably greater than in the downstream channel (see Fig. 1). It was realized that if the ventilation were insufficient, a low-pressure area would be formed under the fall as air bubbles were trapped and carried downstream, thus causing the water level behind the fall to rise. However, an investigation showed that the ventilation was sufficient to maintain atmospheric pressure behind the fall, and indicated that the rise was due to some other cause. A. Schoklitsch[5] suggests that the difference in level can be accounted for by the change of momentum as the falling jet is deflected by the bottom.

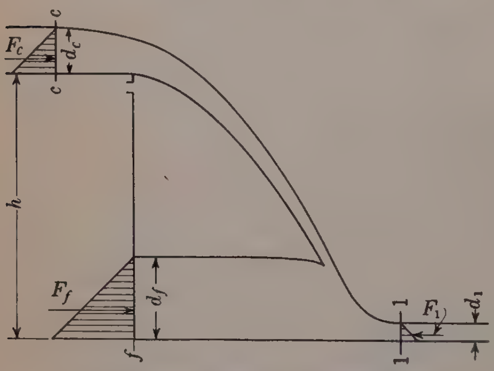


FIG. 8.—OVERFALL, SHOWING FORCES ACTING

Application of the momentum principle to the falling jet requires a knowledge of its profile and of the velocities attained. This difficulty was avoided by dealing separately with the horizontal and vertical components of momentum. Since the vertical force component resulting from the change in the vertical momentum must be taken by the bottom, the horizontal change of momentum may be dealt with in a manner similar to that used for the hydraulic jump. Referring to Fig. 8 and neglecting friction, the momentum equation gives

$$F_c + F_f - F_1 = \frac{\gamma}{g} q (V_1 - V_c) \dots\dots\dots\dots (7a)$$

in which the terms on the left side of the equation represent the hydrostatic forces and the right side gives the rate of change of momentum between sections *c–c* and 1–1. The hydrostatic forces are given by

$$F_c = \frac{\gamma\, d_c^2}{2}; \qquad F_f = \frac{\gamma\, d_f^2}{2}; \qquad \text{and} \qquad F_1 = \frac{\gamma\, d_1^2}{2} \dots\dots (7b)$$

in which d_f is the depth of standing water behind the fall. Substituting Eq. 7*b* in Eq. 7a, introducing the continuity equation, and solving for $\left(\frac{d_f}{d_c}\right)^2$:

---

[5] "Hydraulic Structures," by A. Schoklitsch (translated by Samuel Shulits), A. S. M. E., Vol. I, 1937, pp. 118–119.

$$\left(\frac{d_f}{d_c}\right)^2 = \left(\frac{d_1}{d_c}\right)^2 + 2\left(\frac{d_c}{d_1}\right) - 3 \ldots\ldots\ldots\ldots\ldots (8)$$

In attempting to check this relation against experimental values, difficulties were encountered because it was not possible to measure d_1 accurately for the reasons mentioned previously in dealing with losses at the base of a fall. To overcome this difficulty a value of d_1 was calculated from the experimental value of E_1, using Eq. 5, or:

$$\frac{E_1}{d_c} = \frac{d_1}{d_c} + \frac{1}{2}\left(\frac{d_c}{d_1}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Inasmuch as E_1 was measured to an accuracy of ± 5%, it was felt that this method gave a more probable average value for d_1 than could be obtained with a point gage in spite of the fact that the method of calculation assumed an artificial condition of uniform velocity at the base of a fall. The value of d_f was obtained accurately from the piezometers. In Fig. 9(a) a plot of the momentum Eq. 8 is compared with experimental values of $\frac{d_1}{d_c}$ and $\frac{d_f}{d_c}$. This curve demonstrates that the presence of standing water behind the fall can be explained by the change in horizontal momentum as the water strikes the channel floor. A knowledge of the presence of standing water behind the fall and of its depth should be of value in making stability calculations of a drop structure.

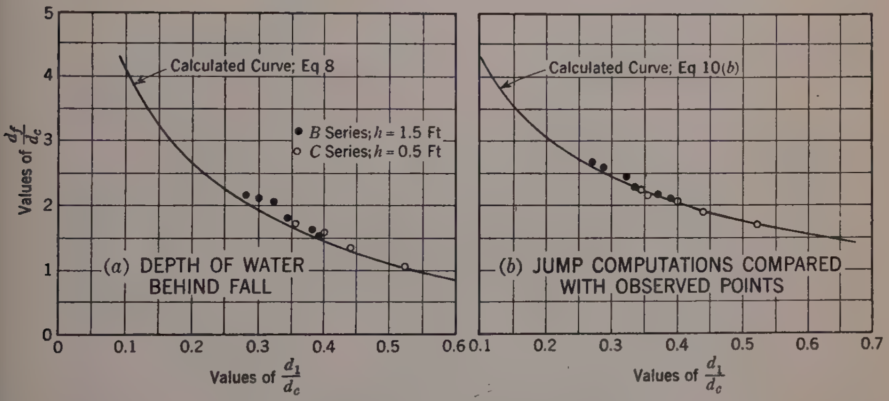


FIG. 9.—CURVES OF MOMENTUM EQUATIONS

FLOW CONDITIONS WITH TRANQUIL TAILWATER

The second part of the investigation checked the vertical elements of the jump below a fall against the momentum equation, and included an investigation of the longitudinal characteristics of the flow as influenced by the tailwater.

A series of runs was made with the tailwater adjusted to form a hydraulic jump at the toe of the fall. In addition to determining the surface profile and bottom pressure throughout the length of the jump, velocity profiles were taken on the center line of the channel. A good indication of the variation in velocity energy along the high-velocity jet that undershoots the "roller" of the jump was obtained from integrations based upon the velocity profiles. Fig. 10 presents a sample plot of these data, in which (similarly to Fig. 6):

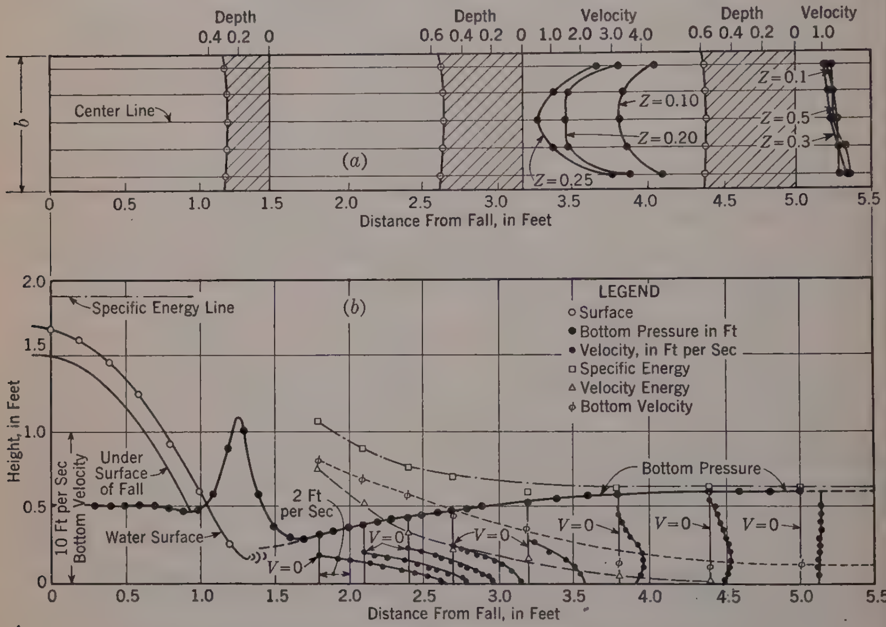


Fig. 10.—Typical Plot of Data for a Jump at the Toe of the Fall

Z = the elevation of the plane of the horizontal velocity traverse; $h_v = 1.55$ ft of water; $Q = 0.664$ cu ft per sec; $d_c = 0.255$ ft; $d_0 = 0.182$ ft; $d = 0.61$ at 7.0 ft from the fall; and $\dfrac{\sum(V^2\,\Delta A)}{2\,g\sum(V\,\Delta A)} + d$ is the specific energy. Longitudinal variations of the velocity head and the specific energy head are shown by the dash and dot-dash lines, respectively. It is interesting to note how rapidly energy is lost near the start of the jump. The dotted line shows the way in which the bottom velocity decreases in the downstream direction. The boundary of the undershooting jet was taken as the zero velocity point of the velocity profiles in the region of the jump.

Vertical Elements of Jump.—The vertical elements of the jump checked very closely with the momentum equation given by Professor Bakhmeteff:[6]

$$\frac{d_2}{d_1} = \frac{1}{2}\left[-1 + \sqrt{1 + 8\left(\frac{d_c}{d_1}\right)^3}\,\right] \ldots\ldots\ldots\ldots\ldots (10a)$$

---

[6] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., 1932, p. 241.

or

$$\frac{d_2}{d_c} = \frac{d_1}{2\,d_c}\left[-1 + \sqrt{1 + 8\left(\frac{d_c}{d_1}\right)^3}\,\right] \dots\dots\dots\dots\dots (10b)$$

in which d_1 is the depth of water at the toe of the fall before the jump and d_2 is the depth of water after the jump.

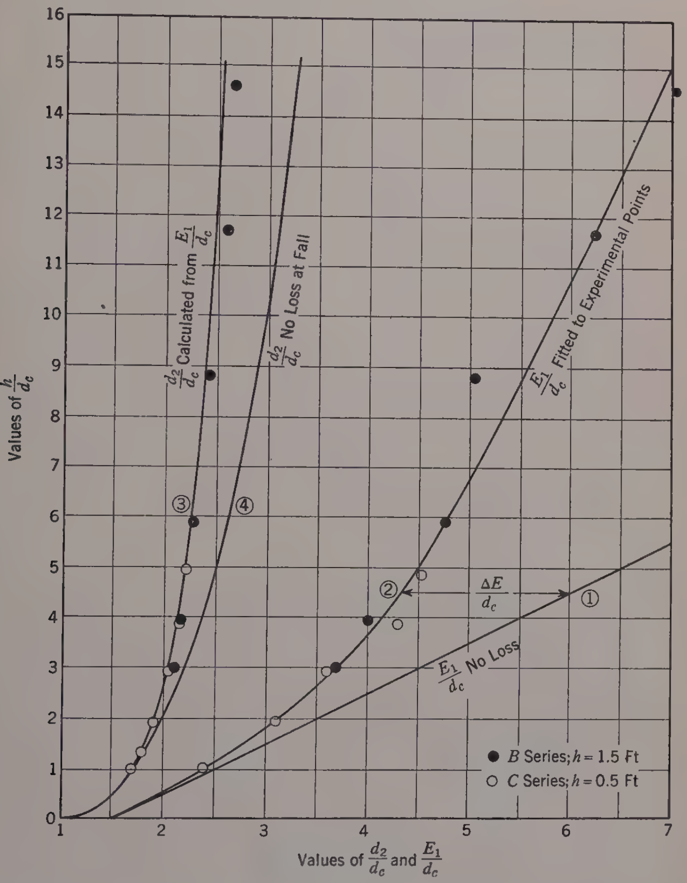


Fig. 11.—Computed and Observed Values of Tailwater Depth Required to Form the Jump

The ratio $\frac{d_2}{d_c}$ was computed from experimental values of d_2 taken with a point gage, and $\frac{d_1}{d_c}$ was computed from experimental values of $\frac{E_1}{d_c}$ using Eq. 9. Fig. 9(b) shows that the values obtained from the momentum equation are in good agreement with those obtained experimentally. This indicates that $\frac{d_1}{d_c}$ computed from $\frac{E_1}{d_c}$ may be used in the calculation of the hydraulic jump.

It is now possible to combine the experimentally determined energy loss with the standard hydraulic jump equation to calculate the minimum depth of tailwater required to form the jump. The values of $\frac{E_1}{d_c}$ may be taken from curve (2), Fig. 7(*a*), and then the value of $\frac{d_2}{d_c}$ may be computed using Eqs. 9 and 10*b*. These computed values are plotted as curve (3), Fig. 11. Curves (1) and (2) are replotted from Fig. 6. The plotted points on curve (3) are experimental values of $\frac{d_2}{d_c}$ obtained with the point gage, and they are in good agreement with the calculated curve. The question arises as to how closely they agree with those computed on the assumption of no energy loss at the base of the fall. The results of such a computation, presented as curve (4), Fig. 11, show that for low relative fall height, $\frac{h}{d_c}$, the agreement is very good. As the relative fall height is increased, however, the curves diverge until at $\frac{h}{d_c} = 12$, the no-loss curve (4) is about 20% above, and curve (3) is about 5% below, the experimental value.

Since as much as two thirds of the energy liberated at the fall may be dissipated as the jet strikes the channel floor, it is surprising that the calculated curve, which assumes no loss at the fall, agrees so closely with the one obtained when all losses are considered. The answer to this apparent inconsistency lies in the fact that the value of $\frac{d_2}{d_c}$ is quite insensitive to the specific energy of the stream entering the hydraulic jump. Although the height of jump d_j will be increased by an increase in the specific energy of the entering stream, the initial depth d_1 is decreased and the resulting value of d_2, being equal to $d_j + d_1$, is only slightly increased. This relation is expressed by the curves of Fig. 12, in which the ordinate shows values of $\frac{E_2}{d_c}$ and $\frac{d_2}{d_c}$ after a jump and the abscissa represents values of $\frac{E_1}{d_c}$. A scale of $\left(\frac{d_c}{d_1}\right)^3 = \lambda_1$ (in which λ is the kinetic flow factor at section 1–1) is included for those who are familiar with Bakhmeteff's kinetic flow factor. It will be noted that $\frac{E_1}{d_c}$ may vary over a considerable range with only a small change in $\frac{d_2}{d_c}$.

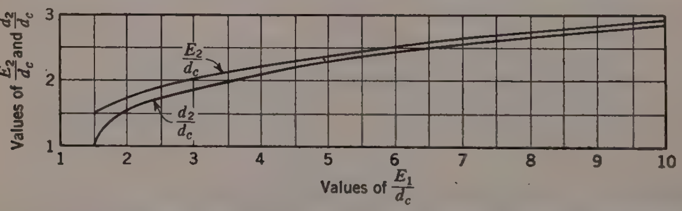


FIG. 12.—DEPTH AND ENERGY AFTER THE JUMP, PLOTTED AGAINST ENERGY BEFORE THE JUMP

When the energy loss at the base of the fall is neglected, the jump calculation may give a tailwater depth as much as 20% high. In some instances this will provide a desirable factor of safety against the jump being washed downstream. Any additional safety factor may cause submergence and impair the effectiveness of the jump as an energy dissipator.

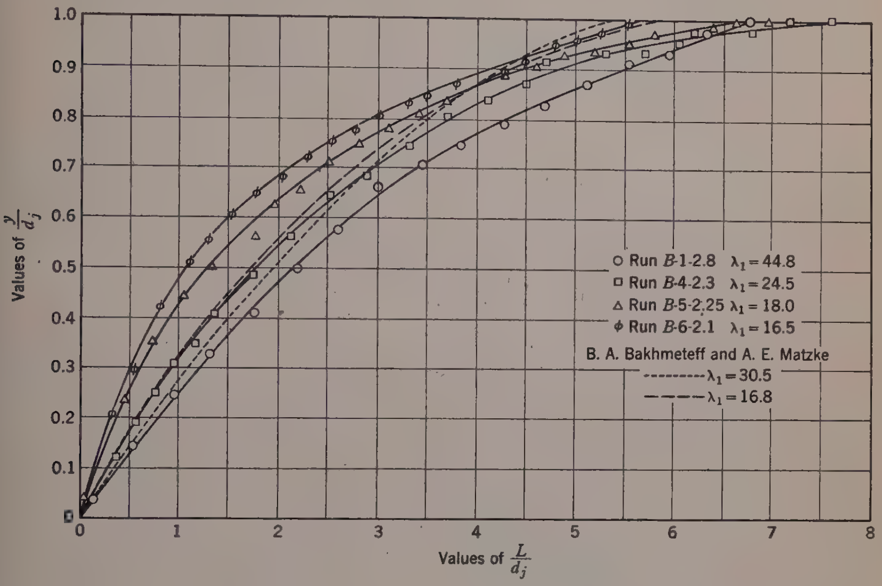


FIG. 13.—LONGITUDINAL PROFILES OF JUMP

Longitudinal Elements of Jump.—A comparison was made of the longitudinal profile of the jump with the results given by Professor Bakhmeteff and A. E. Matzke,[7] Jun. Am. Soc. C. E., as shown in Fig. 13. Since the bottom piezometers gave a good means of averaging the large fluctuations that occur in the surface, they were used to indicate the profile of the jump except at its very beginning where a few point-gage readings were used. Because of non-hydrostatic pressure distribution, the piezometers have a tendency to read slightly high in this region of rapid expansion. This may account for the fact that the profiles rise more rapidly at the start than do those presented by Messrs. Bakhmeteff and Matzke. The profiles given by the writer indicate that below a free overfall the jump is about 20% longer than is shown by the curves of Messrs. Bakhmeteff and Matzke. Since their jump was formed downstream from a regulating sluice, lack of agreement may be caused by a difference in the velocity profile of the shooting flow entering the jump.

The velocity measurements and the energy summations show that the kinetic energy in the undershooting jet is dissipated by the time the surface profile indicates maximum depth. This means that the end of the jump as indicated by the surface profile marks the end of excessive velocities in the undershooting jet. This statement applies to well-formed hydraulic jumps and may well apply to a submerged jump.

---

[7] "The Hydraulic Jump in Terms of Dynamic Similarity," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 630.

Effect of Changing Discharge.—By using curve (3), Fig. 11, it is possible to obtain the minimum tailwater depth necessary to support the jump for any relative fall height. This relation may be combined with the Manning formula to predict the behavior of a two-dimensional simplified drop structure under overload and underload characteristics when the tailwater level is controlled by the slope and roughness of the downstream channel.

The Manning formula for a rectangular section may be written as

$$Q = b\, d_2 \frac{1.486}{n} R^{0.67} S^{0.50}{}_0 \quad \text{.................. (11a)}$$

in which b is width of channel, n is Manning's roughness coefficient, R is hydraulic radius, and S_0 is the slope of the channel bottom.

For two-dimensional flow, $R = d$ and $q = \frac{Q}{b}$; therefore,

$$q = \frac{1.486}{n} d^{1.67}{}_2\, S^{0.50}{}_0 \quad \text{.................... (11b)}$$

Introducing the relation $q^2 = d^3{}_c\, g$ into Eq. 11*b* and rearranging gives

$$d^{1.5}{}_c = \frac{1.486}{n} d^{1.67}{}_2 \sqrt{\frac{S_0}{g}} \quad \text{.................. (12a)}$$

For a given channel the values of S_0 and n will be constant so that, in Eq. 12*a*, $\frac{1.486}{n}\sqrt{\frac{S_0}{g}} = C =$ constant; and

$$d^{1.5}{}_c = C\, d^{1.67}{}_2 \quad \text{.................... (12b)}$$

TABLE 1.—RELATIVE TAILWATER DEPTHS SHOWING OVERLOAD AND UNDERLOAD CHARACTERISTICS FOR TWO STRUCTURES

d_c	Q (%)	DESIGN $\frac{h}{d_c} = 10$			DESIGN $\frac{h}{d_c} = 5$		
		$\frac{h}{d_c}$	$\frac{d_2}{d_c}$	$\frac{d_2}{d_c}$	$\frac{h}{d_c}$	$\frac{d_2}{d_c}$	$\frac{d_2}{d_c}$
(1)	(2)	(3)	(4)[a]	(5)[b]	(6)	(7)[a]	(8)[b]
0.5	35	20.0	2.60	2.57	10.00	2.39	2.34
0.6	46	16.7	2.55	2.53	8.33	2.33	2.29
0.7	59	14.3	2.50	2.48	7.15	2.28	2.26
0.8	71	12.5	2.45	2.46	6.25	2.24	2.23
0.9	85	11.1	2.42	2.42	5.55	2.20	2.20
1.0	100	10.0	2.40	2.40	5.00	2.18	2.18
1.1	115	9.1	2.35	2.38	4.55	2.15	2.16
1.2	131	8.3	2.32	2.35	4.16	2.13	2.14
1.3	148	7.7	2.30	2.34	3.84	2.11	2.13
1.4	165	7.15	2.28	2.32	3.57	2.09	2.11
1.5	184	6.66	2.26	2.30	3.33	2.07	2.09
2.0	283	5.00	2.18	2.24	2.50	1.97	2.03

[a] For the jump. [b] In the channel.

Rearranging Eq. 12b gives

$$d^{-0.1}_c = C^{0.6} \frac{d_2}{d_c} = C' \frac{d_2}{d_c} \qquad \text{(12c)}$$

Eq. 12c shows that when the tailwater is governed only by slope and roughness the value of $\frac{d_2}{d_c}$ will vary inversely as the tenth root of the critical depth. The variation of $\frac{d_2}{d_c}$ required to form a jump may be taken from curve (3), Fig. 11.

Table 1 gives relative tailwater depths, $\frac{d_2}{d_c}$, for two free overfall structures discharging various proportions of their design capacity, Q. At their design flows these structures have relative heights, $\frac{h}{d_c}$, of 10 and 5, respectively. Cols. 4 and 7, Table 1, show the tailwater required to form a jump according to curve (3), Fig. 11, and Cols. 5 and 8 give the depth as regulated by the slope and roughness of the downstream channel calculated according to Eq. 12c, which is based on the Manning formula. The results show that, as the discharge exceeds the design flow Q, the tailwater as regulated by the downstream channel will tend to exceed that necessary to form a jump. When the discharge is less than the design flow, the opposite tendency is indicated. Note that the differences between the values of $\frac{d_2}{d_c}$ required to support the jump and those supplied by the channel are very small. Therefore, if the design tailwater depth is chosen a little high, the structure should perform satisfactorily over a wide range of discharges.

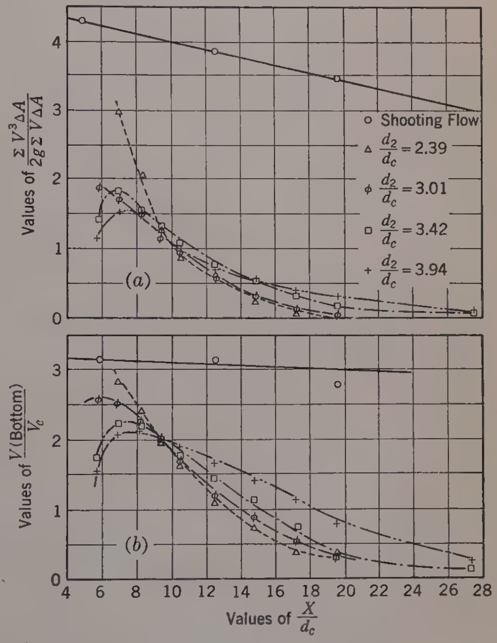


FIG. 14.—EFFECT OF SUBMERGENCE

Effect of Submergence.—An investigation was made to determine the effects of increasing the tailwater until the jump was completely submerged. Data similar to those shown in Fig. 10 were obtained for a single fall height ratio $\left(\frac{h}{d_c} = 5.88\right)$, with four tailwater depths $\left(\frac{d_2}{d_c} = 2.39, 3.01, 3.42, \text{ and } 3.94\right)$. The first of these was just sufficient to hold the jump at the base of the fall and agrees with the value given by curve (3), Fig. 11—that is, for $\frac{h}{d_c} = 5.88$, $\frac{d_2}{d_c} = 2.3$. The higher values caused increasing submergence. By plotting the velocity energy in the high-velocity jet against the distance downstream as in Fig. 14(*a*), it was found that increasing submergence caused the high-velocity jet to persist for a greater distance. A plot of the bottom velocities (Fig. 14(*b*)) also showed higher values downstream as the tailwater depth was increased. This would indicate that oversubmergence of the jump may require bottom protection to be extended farther down the channel. It is felt that further study should be made of the effects of submergence on the rate of decay of the undershooting jet.

"DISSIPATED ENERGY"

It will be noted that in the energy summations made in connection with this investigation only elevation, pressure, and axial velocity energy were considered. Two other forms, turbulence and heat energy, could not be measured.

Methods of measuring turbulence energy have not yet been developed to the point where they are practicable for standard laboratory use. It was not possible to measure the increase in heat energy from the rise in water temperature, because of the minute variations involved. For instance, the complete conversion of 0.01 ft of head to heat would cause a temperature rise of the order of 1.3 times 10^{-5} degrees Fahrenheit. If the turbulence and heat energies had been measured, they would undoubtedly account for the so-called "dissipated energy." It is regretted that this "dissipated energy" could not be separated into its component parts. The heat energy is truly dissipated in the sense that it is no longer available to scour the channel, but it has been demonstrated that the turbulence energy is available for the entrainment and suspension of sediment.

CONCLUSION

This investigation has only started to solve the problems that are involved in the design of a drop structure of the type used in the field. The findings of this study may be summarized as follows:

(1) The loss of energy head at the base of the fall is neither negligible nor constant, but varies over a considerable range as a function of relative fall height (see Fig. 7(*a*)).

(2) The depth of water standing behind the fall is greater than the depth of water shooting from the base of the fall and may be computed from the momentum principle.

(3) The momentum equation for the hydraulic jump may be applied to the flow conditions at the base of the fall to determine the vertical elements of the jump.

(4) The longitudinal profile of the jump is in fair agreement with the profile obtained by Messrs. Bakhmeteff and Matzke. The length of jump below a fall appears to be about 20% longer than the jump below a sluice.

(5) The velocity energy of the undershooting jet is spent by the time the surface profile indicates the end of the jump.

(6) Oversubmergence of the jump will cause the undershooting jet to persist for a greater distance downstream.

(7) The free overfall two-dimensional drop structure will perform satisfactorily over a wide range of discharges.

This investigation has provided information sufficient to enable the hydraulic engineer to produce a rational design for a two-dimensional drop structure, the performance of which can be predicted accurately. Probably it is even more important that the designer can now obtain at least a first approximation of the performance of those more complicated structures that circumstances usually necessitate.

ACKNOWLEDGMENTS

Generous help from members of the staff of the Cooperative Laboratory, California Institute of Technology, is gratefully acknowledged. Ralph W. Koehnke deserves particular recognition for his assistance in collecting data. The work was done with the aid of Work Projects Administration (Official Project No. 65-2-07-58, Work Project No. PS-11496).

APPENDIX

NOTATION

The following letter symbols, used in the paper, conform essentially to American Standard Letter Symbols for Hydraulics, prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1941:[8]

A = cross-sectional area of flow; ΔA = element of area used in discharge and energy summations;

b = width of channel;

C = (C and C' are constants in Eqs. 12);

d = depth of water:

d_0 = depth of flow at the crest of the overfall—that is, at section 0–0;

d_1 = depth of water at the toe of the fall before the jump—that is, at section 1–1;

d_2 = depth of water after the jump—that is, at section 2–2;

d_c = hydrostatic critical depth = $\sqrt[3]{\frac{q^2}{g}}$;

d_f = depth of standing water behind fall;

d_j = height of jump = $d_2 - d_1$;

---

[8] Publication pending.

E = total energy head:

E_0, at the crest of the fall (section 0–0);

E_1, at the toe of the fall (section 1–1);

E_2, after the jump (section 2–2);

ΔE = loss from the crest to the toe of the fall;

F = hydrostatic force; F_1, F_c, F_f = forces at sections 1–1, c–c, and f, respectively (see Fig. 8);

g = acceleration of gravity = 32.2 ft per sec per sec;

h = height of the fall;

h_v = head across venturi meter;

L = length coordinate of the jump measured;

M = from the beginning of the jump momentum; ΔM = change in horizontal momentum;

n = roughness factor in the Manning formula;

Q = discharge in cubic feet per second;

q = discharge in cubic feet per second per foot of width;

R = hydraulic radius;

S = slope; S_0 = slope of channel bottom;

V = velocity:

V_1 = velocity at the base of the fall (section 1–1);

V_c = critical velocity = $\sqrt{g\, d_c}$;

X = distance downstream from the fall, in feet;

y = height coordinate of the jump, measured above the initial depth d_1;

Z = elevation of the plane of the horizontal velocity traverse;

γ = specific weight of water (weight per unit volume);

λ = kinetic flow factor $= \frac{V^2}{g\,d} = \left(\frac{d_c}{d}\right)^3$;

ϕ = velocity coefficient to account for losses.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

FUNDAMENTAL ASPECTS OF THE DEPRECIATION PROBLEM

A SYMPOSIUM

Discussion

BY MESSRS. EDWIN F. WENDT AND L. T. FLEMING, AND ANSON MARSTON

EDWIN F. WENDT,[11] M. AM. SOC. C. E., AND L. T. FLEMING,[12] ESQ.[12a]—In 1911 the Board of Direction appointed a Special Committee to Formulate Principles and Methods for the Valuation of Railroad Property and Other Public Utilities composed of the following eminent engineers, all deceased: Frederic P. Stearns, Chairman, Leonard Metcalf, Secretary, Thomas H. Johnson, Alfred Noble, William G. Raymond, and Jonathan P. Snow, Members, Am. Soc. C. E. In 1914 Henry E. Riggs, Past-President, Am. Soc. C. E., William J. Wilgus, Hon. M. Am. Soc. C. E., and the late Charles S. Churchill, M. Am. Soc. C. E., were appointed to the committee to fill vacancies. The committee reported on every phase of the appraisal of public utilities, outlining the principles and discussing the fundamental aspects of the depreciation problem. Their report of 1917[3] was a classic of its time and so far as fundamentals are concerned there has been no significant change in the succeeding twenty-five years.

However, developments in procedure have resulted from the activities of the many regulatory agencies concerned with the subject. The Interstate Commerce Commission, charged by the Valuation Act of 1913 with the duty of making valuations of the railroads and other common carriers, has given considerable time and thought to the depreciation question, and its record on that subject consists of more than 10,000 pages of testimony and argument. The Commission appointed William D. Pence, John S. Worley, Edwin F. Wendt, and the late Robert A. Thompson and the late Howard M. Jones, Members, Am. Soc. C. E., to an Engineering Board which outlined the procedure that was followed in the field and office. During the progress of this work on the rail-

---

NOTE.—This Symposium appears on pp. 1611–1637 of this issue of *Proceedings*.

[11] Cons. Engr., Pittsburgh, Pa.

[12] Cons. Engr., Chicago, Ill.

[12a] Received by the Secretary October 17, 1941.

[3] *Transactions*, Am. Soc. C. E., Vol. LXXXI (December), 1917, p. 1311.

road valuations, from 1913 to 1921, there was developed what is known as the federal method of valuation, which is now very generally followed in public utility valuation work.

The 1917 report of the Valuation Committee[3] gave exhaustive treatment to depreciation theory and its application in the field of railroad and public utility valuation. This Symposium extends that report, in its fundamental aspects, to new fields—competitive industry, and public works and government finance.

As the leading thesis of his paper, Mr. Walls compares the determination of the depreciation in property as of any time by the method, which he favors, of using engineering and technical knowledge in the same way as it is used in connection with safety standards, replacements of property, and selections of equipment, and by methods based on age-life theory. A question is raised as to the correctness of the conclusion that future life of public utility property can be estimated on an actuarial or statistical basis in the manner that insurance companies treat human lives, with the possible exception of communication facilities and a relatively small proportion of other utilities.

Most valuations of regulated industry which require estimates of depreciation are for rate base determinations. Here, the broad concept of constitutional value based on the expectation of income from property and largely independent of the depreciation in the property as of the date of valuation must be disregarded. The rate base is an artificial element—it is not necessarily, or often, the value of the property, and the principal problem in these cases is the determination of fair return, toward which the estimation of existing depreciation is but a part. Fair return means what is right and just to all parties, and it can never be more than an approximation to the ideal. Throughout the life of each element of a property, the ideal would provide a uniform rate of return on the unamortized part of the investment, uniform annual depreciation charges, and a uniform total annual cost of return, depreciation, and operating expenses, which is manifestly an impossibility to achieve.

The estimates of depreciation must not be subject to unreasonable error, and it is highly desirable that the cost of making the estimates be kept to a minimum consistent with the necessary accuracy, and also that the method of estimation used be not too complex for clear exposition before commissions and courts. The age-life method is advantageous in this respect, especially in the case of properties of considerable size in the valuation of which the engineer who is to introduce and support the valuation can do but a small part of the work. Furthermore, age-life estimates are required for the purpose of setting, or substantiating, the estimated future annual depreciation charges to operating expenses which come ahead of fair return; and it is difficult to refute the argument that the same age and life estimates should determine the depreciation existing at the date of valuation.

The Interstate Commerce Commission has made valuations, for rate-making purposes, of the properties of more than 1,800 steam railroads, most of the field work being done between 1913 and 1922, inclusive. Depreciation was estimated on the straight-line basis, life expectancies being based, in general, on established average lives for the different types of property and not on a true actuarial basis. Although no publication has been made of a comparison of the actual subsequent mortality of the railroad property inventoried with the estimates of the Bureau of Valuation, it is believed that the estimates have not been discredited by the actual mortality experience.

Obviously, in estimating the depreciation applicable to a specific item of property as of a given date, all relevant facts pertaining thereto should be given consideration by the appraiser whatever depreciation theory is being followed. Under a mathematical basis using ages and lives, the size of the sample from which the life estimates were derived and the existing and prospective changes in conditions affecting those lives should govern the reliance placed on the experience of the past, whereas under no basis can the estimated remaining life of the unit under consideration be ignored. The experienced and careful appraiser will insist on the application of judgment after an inspection of the property and examination of the replacement and maintenance records, whatever the basis; and he will reach substantially the same result by any method. In short, the thought in Mr. Wall's closing sentence may be succinctly reiterated: There is no substitute for sound judgment.

Professor Grant opens up a subject which heretofore has not been reported on by committees of the Society, and which has had but scant recognition in papers published in *Transactions*. The subject is an important one to the many engineers engaged in industrial work, and the sections on engineering economy studies relative to a proposed new investment, and economy studies for proposed retirements, are equally applicable to regulated industry.

In the matter of the so-called losses on retirements of property depreciated under the group method, while the regulations of the Bureau of Internal Revenue preclude the allowance of such losses, for tax purposes, at the time of the retirement in the case of property retired before the estimated average service life of the group has been reached, such losses may often be allowable as deductions when the last unit of the group is retired, if the investment in the group has not been made up through the depreciation charges.

Although Professor Grant undoubtedly is correct in his opinion that, in general, the decrease in value of manufacturing equipment is greater in the early years of the life of the equipment than in the later years, such decrease in value is not, as he points out earlier in the paper, the same thing as accounting depreciation, and there are valid arguments against reflecting such differences in the rates of depreciation used in determining the annual depreciation charges, although they must be taken into account in estimating product costs and in reaching decisions relative to replacements of equipment.

Professor Grant's paper is excellent and is a permanent contribution to engineering literature.

The paper by Messrs. Crum and Winfrey contains a statement of sound public policy. The engineer is an important person in the preparation of budgets and the spending of public moneys: He must account for the expenditures. Depreciation is an element of cost and provision should be made for proper governmental accounting to aid the administrator and the engineer in their public work. These authors have outlined the elements of the problem and have clearly illustrated how depreciation should be used in the management of public works and enterprises operated by governments. They have proved their thesis under "Summary" that

> "depreciation accountancy does have a place in government accounting procedures and finance. * * * Until financial reports take full account of all items of annual costs, and account for investment in physical properties, they must be regarded as only partial truths. * * * Depreciation is inevitable; a good government must recognize and must budget for it in the same manner as for other annual expense."

Anson Marston,[13] Past-President and Hon. M. Am. Soc. C. E.[13a]—(1) There has been, and still is, much confusion of thought among writers on fundamental depreciation principles. The writer confesses that his own ideas have changed (he hopes progressed) during some thirty odd years of study and considerable writing on this subject, and he reserves the right to make further progress regardless of what he has already written. Perhaps this confession may win him pardon for disagreeing with some of the views expressed in the Symposium papers.

(2) Apparently the confusion of ideas about depreciation, referred to by Mr. Walls and Professor Grant, is due mainly to the seeming inability of most writers on depreciation to realize the fact, and the implications of the fact, that in every one of the many billions of purchases and sales, the world over, each year, the exchange value is determined by the present worth of the forecasted (not the recorded), probable (not the actual), future (not the past or the present) annual operation return values of the services yet to be rendered by the item purchased. The result is confused references to "physical deteriorations," "wear and tear," "obsolescence," "inadequacy," and "supersession" as if they were agencies whose effects on depreciation could be calculated in some mysterious way other than by judging their probable effects upon probable future service lives (expectancies) and upon probable ratios of future to average annual operation return values. There has been a "blind spot" here in thinking about value (including the negative value called depreciation). That this "blind spot" is not entirely a universal affliction is shown by Professor Grant's lucid statements in the "Synopsis" and "Example 4," Element (3), that, "It is the look forward that is relevant in the appraisal; the look backward (at age) is only useful if it helps in the look forward (at life expectancy)."

(3) Their mental "blind spot" as to the vital rôle of forecasts in determining value and depreciation has left most writers on depreciation in much the same situation as the old Roman water supply engineers who were unable to conceive a compound unit of measurement of flow of water that included velocity in addition to the cross section of the stream. So Frontinus, in his famous book (translated by the late Clemens Herschel, Past-President and Hon. M. Am. Soc. C. E.), had a hard time trying to explain why the sum of the cross sections of the aqueduct flows was not equal to the sum of the "quinaria" units of area of deliveries from the aqueducts, just as writers on depreciation become confused about its fundamental aspects.

---

[13] Dean Emeritus of Eng., Iowa State College, Ames, Iowa.

[13a] Received by the Secretary October 20, 1941.

(4) A widespread, unfortunate misconception about the rôle of forecasts in determining depreciation and value is the entirely mistaken idea that if the actual facts of future service lives prove (as they almost always do) to be different from the forecasts, the depreciations and values based on the forecasts were wrong. The truth is that it is not the actual future service lives and services, but their forecasts which determine the true depreciations and the true values at the time the forecasts were made. In estimating values and depreciations of values the engineer is concerned primarily with exchange values, what willing purchasers will pay and willing sellers take, in mediums of exchange or goods, for the units of property bought and sold. The other shades of meaning sometimes given by usage to the term "value" (and discussed by Bonbright in the two volumes to which Professor Grant refers[4]) need consideration in connection with depreciation only as they can be expressed in terms of their relationships to basic exchange values. Tucked away rather inconspicuously on a page in Bonbright's work[14] will be found a quotation from the decision written by the well-beloved late Justice Oliver Wendell Holmes of the United States Supreme Court in an inheritance division case, as follows:

> "The value of property as of a certain date, like all values, as the word is used in law, * * * depends largely on more or less certain prophecies of the future, and the value is no less real at that time if later the prophecy turns out to be false than when it comes out true."[15]

(5) Another widespread, unfortunate misconception about the rôle of forecasts is the idea that once made they should ever afterward be adhered to regardless of actual later service life facts. On the contrary, re-examinations of the then existing property units and the conditions under which they are serving should be repeated and new forecasts made of expectancies and future annual operation return value ratios frequently enough to keep the forecasts constantly corrected to agree with the facts of actual service life. Industrial properties are not inert. They are changing constantly, partly from changes in management policies and business conditions, and partly from retirements, replacements, and additions of physical property units. The replacement items often differ greatly from those retired. The imminence of inadequacy, obsolescence, and supersession functional retirements must always be considered in making the repeated new forecasts. Upon completing each new forecast that conflicts with the preceding, the current depreciated values of the current existing property items, or groups of items, should be taken as new depreciation bases. In this manner, the annual actual depreciations are distributed throughout service lives in accordance with the actual cumulative growth of depreciation.

(6) The writer agrees with Mr. Walls' view (see "Synopsis") that "depreciation in property as of any time is an accomplished fact"; and that, "to determine that fact, engineering and technical knowledge should be used * * * [of course, in the most correct, simple, direct way]."

---

[4] "Valuation of Property," by J. C. Bonbright, McGraw-Hill Book Co., Inc., 1937. See particularly Chapter X.

[14] *Ibid.*, p. 743.

[15] Ithaca Trust Co. vs. United States, 279 U. S. 151 (1929).

(7) These "accomplished fact" depreciations are the same for privately owned and publicly owned properties; they are the same also for public utilities and for private industrial enterprise properties, except in so far as these different classes of property differ as to original costs, costs of capital, and rates of fair return.

(8) It is the writer's view, apparently somewhat different from that of Mr. Walls, that the correct way to apply engineering and technical knowledge in estimating depreciation is that described in Paragraph 5, herein; and that industrial property mortality curves can be of great value in the work. Hundreds of such mortality curves have been prepared from actual mortality data of different kinds of physical property and it has been found that some eighteen or twenty types of curves may be used to represent all classes of these properties. In the technique of using them, these types of curves serve simply as aids to the judgment of forecasters who have examined the units personally.

(9) The writer has been greatly interested in the paper by Professor Grant, especially in his exposition of the economic principles that should govern judgments in deciding when to make retirements of existing industrial property units. The writer concurs almost completely in the views expressed in this exposition.

(10) Concerning concepts of depreciation the writer's own idea of the correct concept is stated in the "Definition of Depreciation" given by Messrs. Crum and Winfrey (see "Value and Depreciation"). It may be termed the "present worth" concept. In the technique of its use, the effects of changes in price levels are provided for by using different depreciation bases, equal respectively to:

(*a*) The original cost new, less salvage value—for customary depreciation accountancy;

(*b*) Fair value new, less salvage value—for the Baltimore Street Railway Case decision,[16] legal public utility depreciation—and for engineering valuation depreciation; and

(*c*) Reproduction cost new, less salvage value—for insurance damage depreciations.

(11) The "condition per cent" multipliers are the same for all three depreciation bases. For Professor Grant's Example 1, the salvage value is 0, the original cost base is $5,000, the reproduction cost base is $6,500; and the "straight line" condition percentage is 80%, the original cost depreciation is $1,000, and the reproduction cost depreciation is $1,300. (The $200 difference from Professor Grant's reproduction cost depreciation is his adopted additional deduction for the less modern design and assumed higher early maintenance costs of the 8-yr-old house.)

(12) The writer indorses almost completely Professor Grant's views about "Economic Studies for Proposed Retirements" and "Depreciation in Pricing Policies." Concerning their application in Example 3, it may be remarked that the longer the expectancy of the existing property "old asset," the less

---

[16] United Railways and Electric Co. of Baltimore vs. West, chairman, *et al.* 280 U. S. 234, 247, 50 Sup. Ct. Rep. 123. U. S. Sup. Ct., Jan. 6, 1930.

certainty there can be that its retirement would be wise on the basis of a calculation involving so many factors that can merely be estimated. The result of the calculation, as Professor Grant stresses, can be changed greatly at will by changing the assumed rate of fair return, or the estimated service life, and especially by changing the estimated "annual disbursements." Professor Grant's calculations show that a 40% decrease in the estimated annual disbursements for the substitute unit will increase the calculated wise cost payment for the substitute 245%. Is it not likely that this situation is a main reason for adopting a management rule that new units substituted for present units "must pay for themselves in a short time"? Nevertheless, such computations as those in Example 3 are better helps to judgment than mere hunches in deciding when retirements are wise. In his own practice, the writer has been restricting the use of wise retirement mathematics to cases in which the life expectancies of the "old assets" are nearly 0 without material disbursements for "extraordinary repairs."

(13) Example 4 furnishes a good illustration of what a little difference of opinion about the "annual disbursements" on a substitute unit would do to a valuation of the "old asset" on the basis of wise retirement mathematics. Decreasing the estimated annual disbursements on the substitute unit only 10% cuts the appraisal value of the old asset 53% (from $663 to $312).

(14) The writer finds himself unable to agree with Professor Grant that it would be feasible, or desirable, or even correct, to substitute wise retirement mathematics for the present well-tried and established methods of making engineering valuations of industrial properties. The trend of progress in valuation methods is in a quite different direction. The courts, federal tax officials, and valuation engineers are shying more and more away from theoretical formulas to the solid bases of: (*a*) Continuous-inventory property ledger accounts of the actual original costs of all physical property units (or groups of units); (*b*) total and annual depreciation allowances that check closely with complete mortality records of physical property retirements, and which create and maintain depreciation reserves just equal to the total actual depreciation; (*c*) infrequent changes in the values new and corresponding depreciations not of the total but of individual old property units to allow for the effects in present values of changes in price levels; (*d*) inclusion of only actually disbursed overhead capital-cost expenditures on the values, new; and (*e*) only very conservative allowances for intangible values.

(15) Although the writer is in accord almost completely with the ideas expressed in the Symposium paper on "Relationship to Public Works and Governmental Finance" of the fundamental aspects of the depreciation problem, he rules himself disqualified, for obvious reasons, to comment on this able paper by his former students and present co-workers, Director Crum and Professor Winfrey.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ANALYSIS OF STATICALLY INDETERMINATE TRUSSED STRUCTURES BY SUCCESSIVE APPROXIMATIONS

Discussion

By O. T. Voodhigula, Jun. Am. Soc. C. E.

O. T. Voodhigula,[17] Jun. Am. Soc. C. E. (by letter).[17a]—Professor Castleman points out the advantage of choosing members as redundants in the case of a continuous truss. It is obvious that much work can be saved in computing the carry-over factors and the single redundants, whereas in the case of the reaction redundants every member must be included in every computation.

The writer has no objection to Professor Ellis' preference for "Williot Strain Equations." All the methods used in analyzing statically indeterminate structures are fundamentally the method of distortions or deflections.

Mr. Peery's discussion is very interesting. The writer fully agrees with his statement that "It * * * is more important to select the redundants in such a manner as to simplify Eqs. 3 than it is to simplify the actual solution of the equations." In the example chosen by him, it is possible to obtain a direct solution, because the redundants act at the elastic center, where a redundant has no effect on the others; in other words, the carry-over factors are zero.

Dean Grinter's comments are very pleasing.

Note.—This paper by O. T. Voodhigula, Jun. Am. Soc. C. E., was published in January, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1941, by Francis L. Castleman, Jr., Assoc. M. Am. Soc. C. E.; April, 1941, by Messrs. Charles A. Ellis, and David J. Peery; and May, 1941, by L. E. Grinter, M. Am. Soc. C. E.

[17] Municipal and Public Works, Bangkok, Thailand.

[17a] Received by the Secretary September 29, 1941.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

STABILITY OF GRANULAR MATERIALS

Discussion

BY R. G. HENNES, ASSOC. M. AM. SOC. C. E.

R. G. HENNES,[4] ASSOC. M. AM. SOC. C. E.[4a]—In this paper the authors have advanced the development of soil mechanics and have, perhaps, somewhat shifted the course of that development. All tools available to the engineer for the design or analysis of earthwork and foundations are notably of limited application. Each new tool supplements rather than supplants the older. In the opinion of the writer stability analysis by summation of forces along a curved potential slip surface will remain the more convenient and more reliable method for estimating the stability of earth slopes. Quite possibly, the accuracy of the conventional procedure can be improved by determining the distribution of normal pressure along the slip curve by the proposed method of analysis, instead of by assuming a vertical pressure at each point proportional to the height of the overlying column of earth.

In studying the stability of the foundation of embankments the new method offers opportunity for improvement over solutions that depend upon formulas based on the assumption of a semi-infinite solid, by providing a better approximation of stress distribution over the base of the embankment.

The authors state (see "Introduction") "that the terms 'elastic' and 'plastic' are used herein to describe zones * * * and not as descriptions of types of material used in the structure." There is danger that the casual reader might construe this statement to mean that the validity of the analysis is independent of the elastic properties of the material, whereas actually such is not the case. The authors have not freed the soil analyst from the necessity of pretending that sand and a gravel follow Hooke's law. This they recognize elsewhere in the paper. The assumption that stress is proportional to strain, of course, is common to all the usually accepted equations for stress distribution; but the error thus introduced is less consequential than in the present case when, as in consolidation studies, the object of the analysis is the computation of relatively small pressure changes that occur at considerable depth due to a surface load-

---

NOTE.—This paper by R. E. Glover, Esq., and F. E. Cornwell, Esq., appears on pp. 1639–1656 of this issue of *Proceedings*.

[4] Associate Prof., Civ. Eng., Univ. of Washington, Seattle, Wash.

[4a] Received by the Secretary October 16, 1941.

ing. In the latter case only a small part of the stress-strain curve is involved, and this short arc might reasonably be replaced by a straight line. On the other hand, by its very nature a stability analysis of an earth dam presupposes the possibility of stress intensities ranging in value from zero at a free boundary to maximum resistance in plastic zones. The curvature of the stress-strain graph for such a stress range in an embankment cannot be ignored without adversely affecting the validity of the analysis.

It is to be hoped that further development of the authors' attack on the problem of stability may introduce a substitute for Eq. 3. It appears to the writer that triaxial compression tests provide grounds for believing such a development at least a possibility. Data from a limited number of triaxial tests on sand performed at the University of Washington at Seattle, Wash., indicate that, for a given lateral pressure, the major part of the axial stress-strain curve can be fairly well described by the equation

$$\epsilon = \zeta^{\frac{\sigma - n}{m}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

in which ϵ represents strain, m and n are constants for the given test, and ζ is the base of natural logarithms. The next step would be to formulate an expression for strain involving both axial and lateral pressures.

Eq. 3 can be obtained by substituting for strain in terms of stress in the equation:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \epsilon_{xy}}{\partial x \, \partial y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

If more thorough investigation should confirm the writer's hypothesis that there is available a better approximation of the facts than Hooke's law, then substitution of the former in Eq. 31 would provide a basis for improving upon Eq. 3, and would possibly lead to more realistic solutions than Eqs. 15.

It is true that such a development, if successful, would increase somewhat the amount of laboratory work prerequisite for mathematical analysis. This in itself would be a healthy situation. The general adoption of more rigorous mathematical methods has a tendency to obscure the dependence of all analysis on the accuracy of the basic physical measurements. Enforced reliance on the laboratory serves to modify overconfidence engendered by mathematical precision, while at the same time it helps to justify use of these same theoretical refinements.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

CAVITATION IN OUTLET CONDUITS OF HIGH DAMS

Discussion

By Harold A. Thomas, M. Am. Soc. C. E., and Emil P. Schuleen, Assoc. M. Am. Soc. C. E.

Harold A. Thomas,[33] M. Am. Soc. C. E., and Emil P. Schuleen,[34] Assoc. M. Am. Soc. C. E. (by letter).[34a]—That so many distinguished engineers have seen fit to contribute to the discussion of this paper is a source of gratification to the writers, and they are glad to acknowledge the constructive nature of the comments and suggestions received in this way. Their appreciation should not be obscured by the argumentative tone of this final closure, which, of necessity, must be devoted largely to answering specific questions or criticisms.

Mr. Fee makes a number of criticisms of the analytical methods of the paper, but the writers cannot concede the validity of any of his arguments. In his discussion of Fig. 5 and Eqs. 1 to 4, Mr. Fee discards the writers' assumption that the slug of water travels with constant kinetic energy and assumes instead that its momentum remains constant. He develops Eqs. 14 to 17 on the latter basis, and, because Eq. 17 does not agree with Eq. 4, he states that the writers' "entire argument must therefore be regarded as unsound." The writers would like to point out that in this problem the assumption of constant momentum is quite inapplicable. In accordance with Newton's laws of motion, the momentum of a mass of material remains constant only if no unbalanced external forces act on the material. In this problem a large unbalanced force acts on the slug of moving water—namely, the upstream component of the pressure exerted on the slug by the walls of the tapering channel. Under the action of this force, the momentum of the slug is quickly reduced to zero. Therefore, to assume that the momentum is constant in this problem is entirely incorrect.

Note.—This paper by Harold A. Thomas, M. Am. Soc. C. E., and Emil P. Schuleen, Assoc. M. Am Soc. C. E., was published in November, 1940, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1941, by Jerome Fee, Assoc. M. Am. Soc. C. E.; April, 1941, by Messrs. V. E. Leman, P. S. O'Shaughnessy and E. S. Randolph, and Carroll F. Merriam; May, 1941, by Hunter Rouse, Assoc. M. Am. Soc. C. E.; June, 1941, by Messrs. G. H. Hickox, and J. M. Mousson; and October, 1941, by J. H. Douma, Jun. Am. Soc. C. E.

[33] Prof., Dept. of Civ. Eng., Carnegie Inst. of Technology, Pittsburgh, Pa.

[34] Senior Engr., U. S. Engr. Office, Pittsburgh, Pa.

[34a] Received by the Secretary October 6, 1941.

To confirm his statement that the writers' assumption of constant kinetic energy is unsound, Mr. Fee includes the following sentences in his discussion:

> "What would be the fate of this slug of water, for example, if the pyramidal section of the narrowing passage, shown in Fig. 5, had suddenly ceased to taper at a certain point, and had continued, instead, as a constant rectangular section of indefinite length? The writer can only imagine that the slug of water would tear itself to pieces in an effort to conform to the assumption that the kinetic energy is constant."

It is not necessary to use imagination to determine that the slug will "tear itself to pieces" under these conditions, as this fact can be readily verified by computation or experiment. An application of the method used in deriving Eqs. 1 to 4 will demonstrate that the pressure in the water of the slug becomes negative just after it enters the smaller rectangular section, thus causing the slug to break into fragments as it enters the smaller section. A simple experimental demonstration of this action may be made with a quart bottle whose form approximates two cylindrical tubes connected by a tapering section. If the bottle is filled about a quarter full of water and is then held mouth upward in the hands and is jerked suddenly downward, the jet issuing from the mouth of the bottle will not be solid but will be broken into spray. The action can be shown to even better advantage in a piece of apparatus consisting of a 1-in. glass tube about 1 ft long, drawn down to a diameter of about $\frac{3}{8}$ in. in its middle third. About an ounce of water may be placed in the tube, and the ends sealed after exhausting the air with a vacuum pump. When this apparatus is jerked back and forth, the disintegration of the slugs of water as they enter the constricted section is clearly apparent. The existence of this phenomenon is a confirmation rather than a contradiction of the correctness of the writers' assumption of constant kinetic energy.

Mr. Fee's discussion contains the following sentences: "Referring next to Table 3, there are two very different phenomena that should be distinguished" —steady continuous flow and impact. "The authors apparently have no difficulty in transferring quantitative results from one case to the other. In fact, they seem to make little distinction between them." The writers feel that the comment contained in these sentences is quite unjustified. The entire phenomenon under discussion is one of impact caused by a detached mass of water striking a stationary solid or liquid. In preparing the condensed column headings for Table 3 and the accompanying explanation, the writers felt that this point would be so thoroughly appreciated by the reader that it would be unnecessary to include specific statements that the data were inapplicable to cases of steady continuous flow.

Mr. Fee's discussion devotes more than a page to a derivation (suggested by the method of Messrs. Diederichs and Pomeroy[10]) of the ordinary formulas for water-hammer pressure in a pipe. The writers do not consider that this material is pertinent to the discussion of the paper or that any point is gained by plotting the results of an ordinary water-hammer computation in Fig. 14. However, they will refer to this derivation again in discussing a criticism of Eq. 6.

---

[10] "The Occurrence and Elimination of Surge or Oscillating Pressures in Discharge Lines from Reciprocating Pumps," by H. Diederichs and W. D. Pomeroy, *Transactions*, A. S. M. E., Vol. 51, 1929, p. 9.

The following sentences are contained in Mr. Fee's discussion:

> "The writer feels certain that the analysis [leading to Eq. 6] is unsound. The conclusion (see paragraph following Eq. 6) that 'the velocity of particles at the contact surface is one half the initial velocity of the slug of water' is * * * based on an idea that is wholly unreal.* * *
>
> "Eq. 6, which the authors intend as a proof of their proposition, is nothing more than an arbitrary statement that assumes, to begin with, that the impinging particles, and those impinged upon, move during a finite interval of time with a common velocity v_c, which is constant."

Perhaps the significance of the analysis leading to Eq. 6 could be made clearer by considering conditions within a closed pipe, rather than those at the stagnation point on the contact surface where an irregular slug of moving water meets a horizontal water surface as in the diagram at the left of Fig. 6. Mr. Fee's derivation of the water-hammer formulas, previously referred to, assumes a still body of water (in a cylinder or pipe) that is suddenly struck by a (rigid) plane surface such as a piston, and leads to the maximum pressure rise given by Eq. 23. Now, instead of this, assume a still body of water in a cylinder or pipe to be suddenly struck by an elastic piston consisting of water. Imagine the contact surface between the water which was originally still and that which was moving to be defined by a thin film or web, say tissue paper. If one applies himself to the solution of this new problem, he will recognize the fundamental nature of the requirement that in each of the two elastic bodies the front of a pressure wave must travel away from the tissue-paper contact surface at the acoustic velocity. Starting from this fact and proceeding by irresistibly logical steps, using Mr. Fee's method or any other of the ordinary derivations of the water-hammer formulas, one is led to the necessary conclusion that (before reflected waves return) the tissue-paper contact surface must move with a velocity equal to half the original velocity of the water piston, exactly as expressed in Eq. 6, and that the pressure rise must be exactly as expressed by Eq. 7. It is unnecessary to repeat the algebraic details of this proof, since the case is precisely similar to that shown in Fig. 6 and explained in connection with it, the vertical lines *AB* being interpreted as representing the walls of the pipe.

Mr. Leman reveals the fact that high pressures can be produced by the sudden checking of the velocity not only of a detached slug of water, but also of a larger mass of water extending continuously to the source, as in the more familiar case of water hammer in an ordinary pipe. Although the writers agree with him fully on this point, they chose a detached slug for demonstration of the fundamental principle of velocity concentration because the mathematical analysis seemed simpler and because close observation of cavitation phenomena under many different experimental conditions seemed to emphasize the importance of the part played by flying fragments of detached or semi-detached water.

Messrs. O'Shaughnessy and Randolph contribute additional data of much interest in connection with the use of fillers for the stop-log slots in the Madden Dam, and the admission of air through vents in the stop-log slot fillers. They offer the comment that "It is not made clear in the paper how the air measured in the model tests * * * can be related quantitatively to the air that would be required in venting the prototype." In reply to this comment, it may be stated that the analysis of the experiments did not attempt to establish such a quantitative relation. Since air entrainment is a function of bubble size, which in turn is primarily dependent on surface tension, it follows that Froude's law is entirely inadequate to relate the percentages of air entrained in the respective waters of a model and its prototype. During the model tests on the Madden conduits, the quantity of air drawn into the vents was measured as a control for repeating experiments, but it was not considered feasible to relate this air flow to that of the prototype.

Mr. Merriam's discussion contributes interesting information on the objectives and methods for cavitation tests on metal in hydraulic machinery as contrasted with the writers' tests on concrete surfaces in the outlet conduits of dams. The writers appreciate Mr. Merriam's clear statement of the fact that

> "In the case of discharge conduits, which may be operated at full capacity but a relatively short period of time, the idea of the designer is to avoid all danger of cavitation, whereas the builder of high-speed hydraulic machinery is seeking the economic balance between operating revenue and cost of repairs. Consequently, he must strive for quantitative measurement * * *. The former is more interested in qualitative tests which give reasonable assurance that the design proving best in the model is the most suitable for the prototype."

Considerable difficulty has been experienced in making this point clear to various engineers visiting the laboratory during the tests described in the paper, or in convincing them that, in the case of cavitation tests on models of fixed concrete structures, it is a waste of time and money to duplicate the elaborate technique used by hydraulic machinery cavitation laboratories for controlling the air and gas content of the water.

Mr. Merriam states that his principal point of disagreement with the writers lies in their not entertaining the theory advanced by Mr. Pagon[18] in 1935. This theory is discussed by the writers with some detail in their original full-length manuscript, which is on file in the Engineering Societies Library.[3] In essence, Mr. Pagon emphasizes the importance of the low-pressure region at the center of an eddy as a primary source of a "vapor void," and attributes a sudden collapse of the tubular vapor void along the eddy axis to dynamic instability of the eddying motion. In condensing the paper for publication, the writers decided to omit the discussion of this theory because, in their opinion, its tube-collapse feature did not seem to offer an adequate or complete explanation of the principal experimental facts in the type of cavitation under consideration. Granting the importance of curvilinear or eddying flow as a source of low pressure for opening up vapor voids, the writers nevertheless believe that in the types of cavitation under discussion the most important cause of the sudden collapse of vapor voids is simply the impact of swiftly moving water containing such voids against less swiftly moving water or solid objects, and that such collapse will occur whether or not the water is eddying.

---

[18] "Cavitation and Erosion Investigated as a Problem in Fluid Mechanics," by W. Watters Pagon (unpublished papers, A. S. M. E., 1935). On file for reference in Engineering Societies Library, 29 West 39th Street, New York, N. Y.

[3] 29 West 39th Street, New York, N. Y.

An important light is thrown on this topic by the analogy (mentioned by Mr. Merriam) between the downstream region of a cavitation pocket where such impacts are most active and the turbulent region in an ordinary hydraulic jump. Moreover, Professor Edgerton's motion pictures[6] of cavitation convey no suggestion of the existence of tubular eddy-voids.

In support of the eddy-tube-collapse theory, Mr. Merriam mentions that

> "* * * there is the observation that in two-dimensional, venturi-shaped passages the damage comes where the axes of the eddies would terminate against the side walls rather than on the profiles, which would be more likely to receive the onslaught of slugs of water."

As mentioned in the paper, in introducing Fig. 12, a similar phenomenon was observed in the tests on models of the Madden conduits, the side-wall pitting being shown, and its cause explained by the words

> "* * * the pit first develops along the edge of the large cavitation pocket, where especially intense impacts are caused by collisions between the swiftly moving water of the streamline filaments and the broken masses of water and vapor hurled against these filaments from the interior of the pocket."

The writers believe that this explanation of side-wall pitting conforms with the physical facts more accurately than that based on the eddy-tube-collapse theory.

The writers appreciate Professor Rouse's commendation of the paper in general. However, they disagree with the reasoning by which he attempts to disprove the analytical and experimental evidence that the essential cause of cavitation damage is the production of very high pressures by the collapse of vapor voids. It does not seem probable that Professor Rouse means to deny the possibility that high local pressures occur, since his discussion contains the sentence: "Indeed, one need only hear the violent shocks which are produced by discharging steam under water to concede the issue that the collapse of such vapor pockets might well lead to exceedingly high local stresses." It seems more likely that he merely intends to take issue with the writers regarding the actual numerical values of the maximum pressures. It is to be noted that the writers have not established any such numerical values, but have simply carried the computation of Table 3 to the maximum value permitted by the best available compression tests, those of Bridgeman.[7]

Professor Rouse bases his criticism of the writers' analytical work on the "artificial * * * nature of the assumption of constant kinetic energy in the specimen slug of water," presumably referring to the neglect of the items of viscosity, vapor pressure, and rotational energy. However, the writers believe that in the types of cavitation encountered in the large stationary structures under consideration, the preponderance of evidence indicates strongly that these neglected items are of secondary importance in comparison with the main item of kinetic energy. The writers' analysis is not intended to establish precise values of maximum pressure, but merely to furnish a qualitative conception of the fundamental cause of pressure concentration. They feel that the assumption of constant kinetic energy is satisfactory for this purpose, and that the "artificial nature" of this assumption is not of sufficient consequence to otherthrow this position.

---

[6] "Applied Fluid Mechanics," by M. P. O'Brien and G. H. Hickox, Members, Am. Soc. C. E., p. 34.

[7] "The Thermodynamic Properties of Water," by P. W. Bridgeman, *Proceedings*, Am. Academy of Arts and Sciences, Vol. 48, 1912, pp. 307–362.

Professor Rouse mentions that the assumption of a moving slug of water in a tapering channel is less elegant than that of a momentary spherical bubble of water vapor within an initially stagnant, incompressible liquid of arbitrary volume. The writers feel that, although the latter would be a reasonable assumption to illustrate the shock caused by discharging a steam bubble under water, it has little pertinence in connection with cavitation of the type under discussion, where the kinetic energy of the flying masses of water is very large in comparison with their potential energy of position with respect to the centers of the adjacent voids.

This point may be explained in more detail as follows: The kinetic energy developed by the collapse of a vacuum bubble in initially stationary water under atmospheric pressure is equal to the volume of the bubble times the atmospheric pressure, the latter corresponding to a water column about 34 ft high. The kinetic energy of a slug of water moving at a given velocity is equal to the volume of the slug times the pressure developed by a column of water of height equal to the velocity head. In the case of flow in an outlet conduit of a dam 350 ft high, the latter column might be about 340 ft high. Professor Edgerton's moving pictures[6] of cavitation in a diverging tube show a succession of vapor bubbles separated by moving slugs of water having approximately the same size as the bubbles. Thus, if cavitation of this kind occurred in the conduits of a dam 350 ft high, the initial velocity of the moving slugs would provide about ten times as much kinetic energy for destructive cavitation effects as would be provided by the release of potential energy during the collapse of the voids from initially static conditions.

Professor Rouse mentions that "high-speed motion pictures by H. Mueller[24] of the collapse of vapor bubbles produced in cavitation would indicate that the bubbles tend to flatten on the downstream side rather than to dwindle concentrically," and that "Conclusions of this nature would tend to invalidate much of the authors' basic hypothesis, as do piezoelectric measurements by J. Ackeret[25] of only moderately high instantaneous stresses in zones of cavitation." The writers have already pointed out that Professor Rouse's conception of cavitation shock as arising from the collapse of spherical vapor bubbles in a static liquid is not applicable to the types of cavitation under consideration. The flattening of the vapor cavities is also observed in Professor Edgerton's photographs.[6] This flattening strongly disproves the spherical-bubble-collapse theory and confirms the writers' thesis that the collapse of a void under typical cavitation conditions occurs when the void is caught and compressed between swiftly moving water on its upstream side and stationary or less swiftly moving water or solid material on its downstream side.

---

[24] "Hydromechanische Probleme des Schiffsantriebs," edited by G. Kempf and E. Foerster, Hamburg, 1932, pp. 311–314.

[25] *Loc. cit.*, p. 239.

With regard to J. Ackeret's piezometric measurements of pressures in cavitation zones, attention may be called to the utter lack of success of elaborate attempts by other experimenters to attain direct measurements of cavitation pressures. This point is mentioned by Mr. Merriam, who states that it should be emphasized that the multiplication of impact leads to tremendously high concentrations of stress over an exceedingly limited area, and that this is well illustrated by the experiments cited in the paper by J. C. Hunsaker,[5] in which ordinary methods of measuring high pressures such as are used in internal combustion engines, gun barrels, etc., failed completely as criteria of cavitation severity. Mr. Merriam also states that the use of thin diaphragms, backed up by perforated plates, proved in vain because they were punctured merely by accidental hits rather than by a widespread blow. The impossibility of obtaining consistent piezometric measurements in, or closely adjacent to, cavitation pockets was amply verified during the writers' experiments on models of the outlet conduits of the Madden Dam and other dams.

The writers do not disagree with Professor Rouse's point that a positive wave must invariably be followed by a negative wave of equal magnitude, and that fatigue due to the resulting repeated stress reversal is of importance in causing the ultimate failure of the material. This fact, together with the fact that the negative pressure is especially important in destroying concrete because the latter is much weaker in tension than in compression, is implied in a paragraph of the paper preceding "Description of Apparatus for Cavitation Tests." Moreover, the writers feel that their main thesis is not jeopardized by Professor Rouse's prediction that the "theory of fatigue through repeated impact of not-too-great magnitude will eventually be more generally accepted" as a cause of cavitation. Experiments to determine the fatigue of metals in indentation tests are apparently not available, but from a knowledge of the great pressure intensity required to produce a perceptible permanent indentation under the Brinnell ball or Rockwell diamond in hardness tests, and from a knowledge of the relationship of the stress-reversal fatigue limit to the ordinary elastic limit in tensile or cross-bending tests, it does not seem at all unlikely that the indentation fatigue limit in the harder metals known to be attacked by cavitation may be at least of the order of the highest values of Table 3. If Professor Rouse's words "not-too-great magnitude" refer to stresses of an order considerably lower than this, the writers would take issue with him. In this connection, the writers are not referring to soft or mild metals of the ordinary structural grade, but to harder materials having an elastic limit of the order of several times that of ordinary structural steel. Since no upper limit has been found to the hardness of metals pitted in accelerated cavitation tests, and in view of the intense forces required to produce any type of permanent displacement among the molecules of the hard metals by either single loading, repeated loading, or successively reversed loading, it seems impossible to escape the conclusion that extremely high pressures on at least pin-point areas can be produced by the action of water under cavitation conditions. Further considerations bearing on this topic will be mentioned in the writers' reply to Mr. Mousson's discussion.

---

[5] "Cavitation Research," by J. C. Hunsaker, *Mechanical Engineering*, Vol. 57, April, 1935, pp. 211–216.

The writers are pleased that Mr. Hickox agrees with their findings and conclusions in so many respects, a considerable number of these conclusions being restated in his discussion. What seems to be an essential disagreement between Mr. Hickox and the writers is stated as follows in his derivation of Eq. 29:

> "It might be assumed from the authors' remarks following Eq. 8*c* that operation of a model in accordance with Froude's law requires that the ratio of absolute pressures in model and prototype be equal to the model scale. This is not correct."

The source of this disagreement apparently results from a difference in the definition used to extend Froude's law to cover phenomena affected by pressures due to other sources than the action of gravity. In the sentences preceding and following Eq. 8*c* the writers have defined Froude's law as stating essentially that the ratios of all heads including vapor pressure heads in the model and prototype must vary as the square of the ratio of the velocities in model and prototype, whereas the definition used by Mr. Hickox omits the words "including vapor pressure heads." The actual physical facts, independently of this definition, are expressed correctly by the writers' equations. The same remarks apply to Mr. Hickox's comments on Eq. 13, the apparent difference of opinion lying merely in the definition used for Froude's law. The discrepancy that Mr. Hickox thinks would occur between the physical results of tests based on Eqs. 12 and 13 does not actually exist, both of these equations being derived from Eqs. 9*c* and 11 by simple algebraic transposition.

It might be argued that the best method of extending the definition of Froude's law to cover phenomena not investigated by Froude himself is a matter of personal opinion. However, the writers feel that their extension of this definition has important advantages from the standpoints of logic and convenience.

Mr. Hickox is not quite in agreement with the writers' description of a cavitation pocket "as a void space filled with flying slugs of water," but regards it rather as "a region in which cavities form, travel with the stream, and finally collapse." The latter description is especially applicable to the cavitation phenomenon that occurs in a gradually diverging tube, as illustrated by Professor Edgerton's photographs.[6] However, the former description is better fitted to the cavitation occurring behind sharp corners or projections. Although the writers feel some sympathy toward Mr. Hickox's statement of the reasons why he dislikes the use of the term "cavitation pocket," they feel constrained to use this term until a better one is suggested. "Cavitation area" is objectionable for the designation of a volumetric space, and "cavitation region" is more properly applied to the entire region in which the phenomenon is occurring, including not only the space actually traversed or occupied by vapor voids, but also that of the surrounding flow pattern.

In the fourth paragraph following Eq. 29 Mr. Hickox states that "It is doubtful if the discharge can be assumed arbitrarily. In order that the model flow pattern shall approximate that of the prototype, there should be at least a semblance of agreement with Froude's law." The writers disagree completely with this statement. Works on hydrodynamics contain numerous illustrations of streamline patterns that are fixed by the physical boundary conditions and are independent of the velocity, that of the ordinary jet from a sharp-edged orifice being a familiar example. In the same paragraph, Mr. Hickox calls attention to the fact that the writers have used a numerical example in which the velocity scale is exactly equal to the square root of the head scale, and states that "One is led to believe that * * * such good agreement could hardly be fortuitous." Since the enclosed tank apparatus was originally designed for the accommodation of models of scale conforming to the foregoing relationship, it was often found convenient to assume a pump discharge that would fit this relation exactly. This had the advantage of making the velocity scale come out in even figures. The discharge used in this particular example was assumed in this way. However, other arbitrary pump settings could have been used as well (and were actually used in many tests) without distorting the flow pattern. In operating the diverging-tube apparatus, great deviations from the Froude's law setting were used normally, since a velocity scale far in excess of the square root of the linear model scale was required to produce the necessary pressure reduction in the critical cavitation regions.

In the third from the last paragraph of his discussion, Mr. Hickox states or implies that, if a model were operated according to the equations given by the writers, it might be possible for the model to indicate cavitation that would not occur in the prototype. The writers would like to emphasize that Mr. Hickox is mistaken on this point. His misunderstanding of their equations in this connection probably arises again from the lack of an accepted generalization of Froude's law, applicable in cases where pressure heads other than those due to gravity are involved. Consider a model of the conduit entrance of a dam on which the head is variable, and suppose this to be operated without gravitational or operational distortion and in accordance with the writers' equations. At low head, there will be no cavitation in the model or prototype. At a certain head, incipience of cavitation will appear at some point in the prototype, and the writers' equations provide that the adjustment of pressures and velocity in the model will cause incipient cavitation to appear in the model at exactly the head corresponding to this prototype head and at exactly the corresponding point. At higher heads, stronger cavitation will appear in both model and prototype at this point and other pockets may possibly open up. There is no possibility for cavitation to occur in the model where it does not appear in the prototype, or vice versa. The analytical reasoning establishing the correctness of the foregoing statement is given in the paper in such detail that it seems unnecessary to repeat it. The sentence on which Mr. Hickox's argument is based is stated in the following words by him:

> "If the model is adjusted to operate in conformity with Froude's law [as defined by Mr. Hickox this does not require vapor pressure head

> scale to be proportional to the square root of the velocity scale], and the absolute pressures in reservoir and conduit barrel are adjusted to conform to the linear scale ratio, the model may indicate cavitation that would not occur in the prototype * * *."

This statement is perfectly ture, but the fallacy of the argument lies in the fact that the words quoted do not state the same thing as the writers' equations.

As stated in the paper, if the model is operated with gravitational or operational distortion, the gage settings to produce incipience of cavitation in the model at exactly the same comparative head and at the same point at which it occurs in the prototype cannot be made to apply to the entire fluid system of the model, but a separate setting must be used to test each point at which it is desired to know whether cavitation would occur in the prototype. For example, Mr. Hickox states that he "would like to know how the methods of the paper were applied to a rotated vertical diametrical section of the entrance of the Hiwassee sluice, for example, which was found to be entirely free from cavitation." The Hiwassee cavitation tests are described in the original manuscript on file in the Engineering Societies Library.[3] Since the entrance bell of this circular conduit is nearly a surface of revolution, the first and most likely point to be tested for possible cavitation was at the conduit roof just inside the entrance, where the general pressure was lower than at the floor, because of shallower submergence, and where the velocity was as high as anywhere else. The gage settings were made according to the equations of the paper and the model showed no cavitation at this point, thus proving that there would be no cavitation at the corresponding point in the prototype. Other suspected points, near the rails carrying the closure bulkhead, etc., were investigated in a similar manner but no cavitation was found. By exaggerating the general cavitation-severity condition in the model, it was easily proved that no doubtful points had been overlooked.

The writers appreciate the constructive criticisms by Mr. Mousson, and in particular they are glad to accept his word of caution regarding the importance of the opening up of fatigue cracks as a feature in the mechanism of cavitation damage to metals. As they have stated in the paragraph preceding Eq. 5, the writers realize that "The law governing the * * * process of pressure multiplication undoubtedly changes when the crevices become so small as to approach molecular dimensions." Since the theoretical investigation of the action taking place in very small crevices is definitely a problem for the molecular physicist rather than for the engineer; since direct experimental measurement of pressures in such crevices appears to be totally lacking; and since Mr. Mousson is able to cite practical cases, where cavitation damage to turbine runners was not increased by the presence of hair cracks, and laboratory tests, where subsurface disintegration occurred without surface cracking, the writers are quite willing to hold in abeyance further speculations regarding cavitation action in fine cracks.

Mr. Mousson joins with Professor Rouse in questioning the probable occurrence of pressures approaching or exceeding the higher values of Table 3, and in believing that cavitation damage to metals can be explained satisfactorily as a fatigue phenomenon due to stresses comparable to the lower values of the table. Because of the failure of the most painstaking experiments to give any inkling of true cavitation pressures, this entire topic lies in the realm of speculation. However, the writers believe that the preponderance of evidence points to the conclusion that the harder metals cannot be destroyed in the absence of local pressures of at least the order of magnitude of the higher values in Table 3. A striking illustration of the high intensity of pressure that can be sustained on a local area of a plane surface of a hard metal without producing a permanent deformation is found in the classic series of tests of the bearing of locomotive wheels on steel rails, reported by the late C. L. Crandall,[35] M. Am. Soc. C. E., and in the discussion by the late J. B. Johnson,[36] M. Am. Soc. C. E. In these tests pressures of 164,000 lb per sq in. produced no permanent set.

After reading Mr. Mousson's discussion, one of the writers made some experiments with the Brinnell-ball apparatus to check the pressure intensity required to produce a perceptible permanent indentation in the surface of a hard metal. The first steel specimen selected at random for this test had a Brinnell hardness number of 524, which is about "file hard." Under a load of 500 kg on the Brinnell ball, a slight but definitely perceptible indentation 1.1 mm in diameter was produced on the surface of this specimen. A little arithmetic shows the average pressure on the contact area to have been just less than 750,000 lb per sq in. It is probable that the maximum pressure intensity at the center of the indentation was materially greater. A similar test was run on a piece of metal from an ordinary structural steel angle, the Brinnell hardness number of this steel being 90.4, and a perceptible indentation of 1.40 mm in diameter being produced by a load of 150 kg on the Brinnell ball. The mean pressure under the latter loading thus was about 139,000 lb per sq in., whereas the maximum at the center of the indentation was probably higher. These values carry an implication that, in view of the enormous local pressures required to produce slight permanent indentations (or permanent displacements of the molecules) on the surfaces of hard metals, it might reasonably be expected that pitting would not take place on such metals unless the pressures developed within the cavitation pocket were at least as high as the larger values of Table 3. Although no cavitation tests have been made on the particular pieces of steel mentioned, experience with such tests indicates definitely that extreme hardness or toughness of the metal is no barrier to ultimate pitting and disintegration under severe cavitation conditions.

Analogy with various types of fatigue tests makes it highly probable that the fatigue limit or repeated pressure intensity required to produce a permanent disintegration of the metal would be at least a large fraction of the pressure producing the first permanent set. Moreover, numerous short-time cavitation tests of metals are on record[18] where fatigue has not had time to become effective and where the first evidences of damage are a few scattered tiny pits or prick-punch indentations due to individual blows. Because of the minuteness and the scattered nature of the individual pits or areas in which the extreme pressures act, the writers do not concur with Mr. Mousson's argument that pressure "values in the upper part of the range [of Table 3] are not likely to occur since damage would be so severe in a matter of minutes as to endanger any structure within the same time interval."

---

[35] "Friction Rollers," by the late C. L. Crandall and A. Marston, *Transactions*, Am. Soc. C. E., Vol. XXXII, July-December, 1894, p. 99.

[36] *Loc. cit.*, p. 270.

Mr. Mousson states that another item of evidence against the theory of local high-pressure points in cavitation regions is the fact that, in tests on metals in cavitation stands, throat velocity has been found to be one of the most important variables with respect to pitting. The writers do not recognize the soundness of this argument, since the rate of pitting would obviously increase with the energy content of the individual blows and their frequency, even if the maximum pressure at the center of each affected spot did not increase. However, in this connection it seems desirable to repeat that the writers' analytical work in Eqs. 1 to 7 and in Table 3 was not intended as a complete analysis of the mechanism by which high-pressure points are developed in cavitation regions, but was intended merely to explain or illustrate one of the most salient features of this mechanism—namely, the tendency toward energy concentration at the final point of a void collapse. It should by no means be thought that the writers regard the neglected items of viscosity, vapor pressure, and rotational energy as so unimportant that the maximum intensity of pressure developed by a vapor pocket collapse is entirely independent of the total quantity of energy in the moving water that is brought to rest.

The discussion by Mr. Douma calls attention to the desirability of establishing methods or criteria, based on free-jet profiles, for designing conduit entrances that will be free from cavitation. The writers are much interested in Mr. Douma's ideas on this topic, since they have done considerable analytical and experimental work along somewhat similar lines (described in some detail in the original manuscript on file in the Engineering Societies Library[3]). The rigid space requirements for publication made it necessary to eliminate somewhat more than three fifths of the original subject matter, the section on the design of entrance bells being one of those deleted. The writers are in full agreement with Mr. Douma on the principle that the nappe of a free jet issuing from a sharp-edged orifice has the ideal shape for a cavitation-free entrance bell, and that entrance bells having solid parts projecting inside the location of this nappe surface are especially liable to cavitation. However, it may be remarked that it is not feasible to use this principle in designing an entrance bell unless the upstream face of the dam in the vicinity of the entrance is a plane normal to the conduit axis. Moreover, entrance bells designed with the "ideal" profile will usually have a considerable factor of safety, since the pressure on all surfaces of the bell is equal to that in the upstream end of the conduit proper, and therefore considerably above atmospheric pressure (if the conduit is level or nearly so), whereas the pressure head required to induce cavitation is about 33 ft below atmospheric pressure. This fact may be utilized to achieve economy in the design of entrance bells for conduits under comparatively low head. In the original manuscript, a table of coordinates for the profile of an ideal entrance bell is given, based on G. Kirchoff's exact equations for the free-jet profile[37] rather than on the ellipse used as an approximation by Mr. Douma. Also, a table is included, based on original experiments, giving the end and side contractions of the jet for conduits of rectangular cross sections having various slenderness ratios.

Corrections for *Transactions:* In November, 1940, *Proceedings*, on page 1625, line 12, change 155.3 ft to 168.3 ft; and in April, 1941, *Proceedings*, on page 659, delete the paragraph beginning with line 20.

---

[37] "Hydrodynamics," by H. Lamb, Cambridge University Press, 6th Ed., 1932.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

HYDRAULICS OF SPRINKLING SYSTEMS FOR IRRIGATION

Discussion

By J. E. Christiansen, Assoc. M. Am. Soc. C. E.

J. E. Christiansen,[24] Assoc. M. Am. Soc. C. E. (by letter).[24a]—The interest shown in this paper is appreciated sincerely. Professor Pillsbury has contributed greatly to the subject, especially with respect to undertree sprinkler systems for orchards. Although the writer fully agrees with Professor Pillsbury's statements in regard to the conditions under which sprinkling is useful, he also believes that sprinkling will be satisfactory on most crops, and under most conditions, provided sufficient water is applied, at the proper rates. Exceptions are found in crops adversely affected by fungi or bacterial diseases; sprinkling may aggravate such trouble or make control more difficult. Grapes, a typical example, are not sprinkled because of mildew. Sprinkling is simply a method of irrigation in which the operator has good control over the water. If he applies the water at such a rate that the soil can absorb it, he can apply as little or as much as he wishes and can secure any depth of penetration desired.

The primary limitation is one of economics—that is, the cost of sprinkling as compared with other methods of irrigation. In general, sprinkling costs more; and unless the results, as measured by increased returns, are better, it cannot be considered economical. Since only 1% or 2% of the total irrigated area in California is sprinkled, evidently most farmers are either satisfied with their present methods, or do not believe that increased returns will compensate for higher costs.

Mr. Bither lists many advantages of sprinkling. An increased return resulting from one or another of these advantages will sometimes more than offset higher costs, but not always. One crop grown extensively in California that thrives under sprinkling is Ladino clover. Being shallow rooted, it requires rather frequent irrigation—that is, every week or 10 days for best results. Most of it is grown on rather poor, shallow, undulating soil, where extensive

---

Note.—This paper by J. E. Christiansen, Assoc. M. Am. Soc. C. E., was published in January, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1941, by Arthur F. Pillsbury, Assoc. M. Am. Soc. C. E.; and June, 1941, by Messrs. Tom A. Bither, Ralph W. Powell, and Harry F. Blaney.

[24] Asst. Irrig. Engr., College of Agriculture, Univ. of California, Davis, Calif.

[24a] Received by the Secretary October 10, 1941.

leveling is objectionable. It is generally irrigated by the border-strip flooding method, and considerable water is wasted. Nevertheless, thousands of acres of clover are grown profitably, mainly for fattening lambs, where the gross return averages about \$25 per acre. This total gross return, however, would scarcely pay the cost of applying water with sprinklers; and so, although the production might be appreciably increased, sprinkling would apparently not be feasible. The economic phases of this subject do not seem pertinent in this paper and are covered in another report.[25]

All the tests illustrated in Figs. 3 to 7 were made on sprinklers that have been replaced by later models. However, many tests on types now generally used indicate that the uniformity of application usually is no better than that produced by the pattern shown in Fig. 3. Furthermore, the new sprinklers are similarly affected by wind and to some extent by low pressure. They do rotate at a fairly steady, uniform rate and cannot be thrown out of adjustment so that they rotate rapidly. The uniformity of application of water by sprinklers is the subject of a paper published early in 1941.[26]

The logarithmic diagram (Fig. 11) for determining pressure losses in portable sprinkler lines is based upon coefficients slightly more conservative than those obtained from the field tests on sprinkler pipe. This diagram was prepared as a guide for designing sprinkler lines with all types of couplings. Although tests on several makes of lines indicate rather small differences in friction factors, some allowance was made for deterioration in carrying capacity caused by increasing roughness of the pipe with age. This allowance was very small as compared with standard engineering practices in designing pipe lines. The assumption that all sprinklers discharge equal quantities of water does not result in higher losses, provided proper friction factors are used and provided the total friction loss is not excessive. The writer recognizes the advantage of tables for the purpose mentioned. In this paper, however, tables would be impracticable because of the space required to cover the many combinations of pipe and sprinkler sizes in use.

In deriving Eq. 19 by the momentum treatment, Professor Powell does not consider the pressure difference on the upstream and downstream sides of the side outlet near the main pipe. Undoubtedly there is a zone of high pressure on the downstream side of the outlet pipe, and a zone of low pressure on the upstream side. Whether or not the net downstream thrust on the outlet pipe is sufficient to harmonize Eqs. 5 and 19 is not known. As Professor Powell states in private correspondence with the writer, the effect of this unequal pressure on the sides of the outlet pipe is to reduce the values of β_2 given in Table 2. It is surprising that such an intriguing problem in hydraulics has not been investigated more completely. The writer was concerned principally with the effect of the pressure recovery on the friction loss in the pipe. The friction factor f, computed on the basis of pressure and elevation difference at two points on a sprinkler line, with differences in velocity neglected, was found to remain rather constant for large differences in Reynolds' number. When corrections were made for the pressure recovery at each outlet, however, the customary relation between f and R was obtained. In other words, the pressure recovery at a side outlet increases the exponent of V to approximately 2 in an empirical formula for flow of water in pipes, the magnitude of the effect depending upon the spacing and number of outlets on the pipe. For this reason the Weisbach formula, with constant value for f of each pipe size, was used as a basis for the friction-loss diagram.

---

[25] "Irrigation by Sprinkling," by J. E. Christiansen, College of Agriculture, Univ. of California, Berkeley, Calif. (bulletin in press).

[26] "The Uniformity of Application of Water by Sprinkler Systems," by J. E. Christiansen, *Agricultural Engineering*, Am. Soc. of Agri. Engrs., Vol. 22, No. 3, March, 1941, pp. 89–92.

The writer thanks Mr. Blaney for emphasizing that sprinkling is not a new practice. Portable systems of the types now common, however, are relatively new; and their use has greatly extended the acreage irrigated by this method. As Mr. Blaney also states, sprinkling does not always increase irrigation efficiencies; many installations are very inefficient, mainly because of poor equipment, poor design of the system, and improper operation. In the writer's opinion, however, it is possible to obtain higher irrigation efficiencies by sprinkling than are generally possible by surface methods, although this is not always accomplished. Farmers seldom understand the hydraulics of sprinkler systems; and often some minor change in equipment or in methods of operation will improve the efficiency of the system materially.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

MISSOURI RIVER SLOPE AND SEDIMENT

Discussion

BY MESSRS. LEO M. ODOM, E. W. LANE, AND ELLIOTT J. DENT

LEO M. ODOM,[14] ASSOC. M. AM. SOC. C. E. (by letter).[14a]—Because of the tremendous expenditures that have been made on the improvement of the Missouri River, it is highly gratifying that the profession has received the benefit of a discussion of the principles upon which the general improvement is based and of the results obtained.

Improvement of the Missouri River for navigation is certainly a task of magnificent proportions. The totals are overwhelmingly immense and the design in practically all of its phases involves forces and actions of Nature, the understanding of which (in spite of all the mathematical gymnastics and lengthy dissertations that an intensely scientific age has produced on the subject) seems almost as remote as it did when General Humphreys commenced his Topographical and Hydrographical Survey in 1850.

In arriving at a solution for river control, as in most of the problems that beset the river engineer, he may utilize any and all of the following approaches: (1) Cut and try; (2) judgment based on observation of similar problems; (3) mathematical formulas; and (4) small-scale models. It seems to the writer that all of the writing in the past few years has been about approaches (3) and (4), and most of the work has been performed by the use of approaches (1) and (2).

The author sets forth the general plan of improvement that has been evolved and gives briefly the methods and procedure by which the general plan is to be accomplished.

It was decided at the beginning of the improvement that, although the river meandered widely in Nature, its average slope was still ample to provide self-cleaning velocities and that the course of improvement for low-water navigation should be restricted, in so far as practicable, to confinement of the low-water flow to a well-defined channel, meanwhile revising the alinement so as to obviate direct attack and crossing bars. These considerations led the

---

NOTE.—This paper by William Whipple, Jr., Assoc. M. Am. Soc. C. E., was published in March, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1941, by Samuel Shulits, Assoc. M. Am. Soc. C. E.

[14] Chf. Engr., Dept. of Public Works, State of Louisiana, Baton Rouge, La.

[14a] Received by the Secretary September 26, 1941.

Engineer Department to propose changing the alinement of the river to a continuous series of sweeping curves. The adoption of this proposal embarked the federal government on a project which, of its kind, is the largest that man has ever attempted. The determination of the controlling radii and central angles of the curves to be used was undoubtedly done in the broad laboratory of Nature itself. All that remained was to fit in these curves as economically as possible, and the general plan was complete. The writer believes that the method of cut and try, combined with judgment, was used practically exclusively in evolving this plan.

The author presents a good summary of the type of material borne by the stream, both as to bed load and suspended sediment. The writer cannot follow his discussion of the mechanics of bed load and suspended load movement. The suspended load to be found in a cross section is dependent on the turbulence to a much greater extent than on the velocity, and, although the velocity may give a rough index of the turbulence, it does not do so necessarily. Furthermore, the movement of the major percentage of the detritus progresses downstream by "trading" and not as a continuous process.[15] With these two facts in mind, the author's discussion and conclusions in some cases seem vague. The detritus content in any cross section would not necessarily indicate that the total quantity is progressing downstream since it is highly probable that a large part of such detritus is in the process of transverse movement.

The author's comparison of the results obtained by the improvement with the answers obtained by formulas is interesting and enlightening. The writer had never seen the first two formulas before and was, therefore, not disturbed to learn that they will not work. As to the Straub formula, it is rather a disappointment to learn that it will not work on the river from observations of which it was derived.

The author's statement that the Straub formula was designed for uniform discharge only is in disagreement with the derivation of the formula given in the "308" report on the Missouri River. It is stated therein that the slope is assumed to remain constant; variation of depth and discharge is not supposed to affect the accuracy of results. It is not exactly clear what effect the stage of the river at which widths were measured could have on results given by the formula as cited, since the formula is intended to give the quantity of sediment transported per second along the stream bed in pounds per unit width of channel.

Two glaring weaknesses that are apparent in most river work today are revealed by this paper. One is the fact that the location and number of continuous observation stations frequently are not suitable for design requirements. Another weakness is in the practice of constructing improvement works without an adequate knowledge of what was occurring in the section of the river prior to the improvements. This practice is indicated in the paper in that the author is forced to compare the action in a 25-mile section of the improved river with a different 10-mile stretch of unimproved river at a different discharge. A comparison such as that is of doubtful value. The only logical way to find out whether improvements are accomplishing the desired end would be to have sufficient data on the improved section taken prior to the improvement to provide a basis for comparison with its condition after improvement. The adoption of a common-sense theory of improvement such as was followed in this case is good practice; but all theories, of whatever origin, bear careful checking in the field. The only way that engineers can be absolutely sure that their improvement works are accomplishing the desired results is to make a detailed "before and after" comparison, on the section of river improved.

---

[15] "Mineral Composition of Sands from Mississippi River Tributaries," by R. Dana Russell and Leo W. Hough, *Proceedings*, Geological Soc. of America, 1937, p. 109.

The author has examined the value of the formula approach to design of river-control works. It would be very enlightening to have a similar examination of the value of small-scale models, since, undoubtedly, they have been used in connection with the design of works on the Missouri River.

E. W. LANE,[16] M. AM. SOC. C. E. (by letter).[16a]—Considering the magnitude of the Missouri River improvement project, very little has been written about it, and hydraulic engineers will be very much indebted to the author for his interesting exposition of the principles underlying it. This improvement probably constitutes the most extensive application of the contraction of streams as a means of deepening them for improving their navigability, a procedure which is one of the oldest in the science of hydraulic engineering. A search, by a hydraulic engineer, of the ancient writings of Egypt, Mesopotamia, and China would probably show earlier writings on the principles involved, but the earliest known to the writer is by Chi-Husn Pan in the latter part of the sixteenth century.[17] Mr. Pan argues at length in favor of confining the Yellow River between embankments and making it deepen its channel by scouring out its bed.

Although the Chinese apparently knew many of the principles of river control before the Europeans, this knowledge has had no influence on the development of the present-day science. So far as is known, the earliest work leading to modern knowledge was that of Guglielmini in the latter part of the seventeenth century. He stated that the greater the quantity of water that a river carries, the less will be its fall, which statement was used as the basic principle of plans for contracting rivers. The subject seems to have been a matter of frequent discussion in Europe before the latter part of the eighteenth century. At this time Paul Frizi and M. Gennete[18] argued in favor of keeping rivers in a single channel rather than having them branch near the mouth, as the single channel would have less fall and cause less flooding.

In the United States the principles involved were the subject of active discussion in connection with the improvement of the Mississippi River nearly a century ago. At that time it was proposed to deepen the Mississippi River for navigation and flood control by preventing the overflow of the banks at high-water time by means of levees. This discussion developed into one of the greatest of engineering controversies in connection with the improvement of the passes of the Mississippi River by the late James B. Eads, F. Am. Soc. C. E., and the confinement principle of improvement was adopted as the basis of the early work of the Mississippi River Commission. Although successfully applied to the improvement of the mouths of the river, it was not a success when applied, as a means of flood control, to the entire river, and was superseded by the "levees only" principle. The success in one case and failure in the other was largely due to the tremendously greater quantities of material involved in the case of the entire river, and the lack of a quantitative analysis of the problem, for, although data for an exact solution is still unavailable, it is possible to show by a computation of the quantities that would be moved and the time required, using data then available, that its application to the entire river would fail. The difference between the quantitative relations given in the paper and the almost entire lack of discussion in quantitative terms during the early Mississippi controversy show that appreciable progress has been made in the science, but the qualitative nature of the author's conclusions indicates that considerable progress is still necessary before a prediction can be made of the time interval required to bring about the desired results.

---

[16] Prof., Hydr. Engr., State Univ. of Iowa, Iowa City, Iowa.

[16a] Received by the Secretary October 4, 1941.

[17] "Review of River Protection," by Chi-Husn Pan (translation by Hsing and Wong).

[18] "An Investigation of the Transportation of Bed Load by Flowing Water," by F. T. Mavis; thesis presented to the Univ. of Illinois, Urbana, Ill., in 1935, in partial fulfilment of the requirements for the degree of Doctor of Philosophy (see p. 110).

It is believed that the author would have clarified somewhat the picture of the results of the contraction works if he had amplified the statement in his "Conclusions" that "In view of the great length of river involved, no appreciable change in mean slope can be expected." In the "Synopsis" one reads the statement that the mean slope of the river will eventually decrease through the operation of the contraction works, and later quantitative relations are given tending to show that the equilibrium slope would be reduced from 0.96 to 0.76 ft per mile, a change of 21%, which is in agreement with the 0.69 ft per mile slope observed in an improved stretch. When these statements are followed in the "Conclusions" of the paper, with no further explanation, by the statement that no appreciable change in mean slope can be expected, it seems probable that some readers of the paper may be confused. If this is the case, perhaps the following discussion will aid in clearing up the picture of what will probably take place.

It appears to the writer that there are two more or less distinct effects of the construction of the contraction works: (1) The change of shape of the channel from a wide shallow one to a narrower and deeper one, and (2) a change of the gradient of the stream as it tends to reach equilibrium under the changed conditions. The first of these will tend to take place relatively rapidly, especially the major part of the total change, but the second will be active for a much longer period, and equilibrium will not be reached for many years even from a practical standpoint, for theoretically equilibrium would never be achieved.

To carry the flow of the river at a width of 725 ft, which formerly flowed over a width of 3,650 ft, without raising the water surface, will require a deepening of the bed. Fig. 11 shows the depths required for various discharges using the slope of 0.96 ft per mile in both cases. The difference in depth at any given discharge that would give the same water surface elevation for these two conditions is indicated by the vertical distance between the two lines.

Thus, to obtain, at bankfull elevation, the same discharge in the contracted as in the uncontracted channel (at which stage the unimproved stream discharged about 80,000 cu ft per sec) would require a deepening of the bottom of about 16.5 − 6.3 = 10.2 ft. Under these conditions the area of the channel at bankfull level would be reduced from 23,000 sq ft for the unimproved channel to 12,000 sq ft for the improved channel, a difference of 11,000 sq ft. The area of excavation would be 725 × 10.2 = 7,400 sq ft and the area of fill (3,650 − 725) 6.3 = 18,400 sq ft. The space that would be filled in with sediment under the improved conditions, if it fills to former bank levels, is therefore 67% more than the volume that it is necessary to excavate in the contracted section. Since the competence of the stream is much greater than required to move the largest bed particles, if the size of the bed material does not increase substantially with depth, it is not unreasonable to expect that considerable of the deepening due to contraction will proceed rapidly, as the material excavated at one point along the channel can be deposited outside the contracted section a short distance downstream. However, the river carries great quantities of material in suspension which will deposit in, and rapidly fill up, the parts of the old channel outside the new channel line. As this filling takes place, the deepening of the contracted channel will slow up, and although the first part of the deepening will probably proceed quite rapidly it may be some time before the final depth is reached.

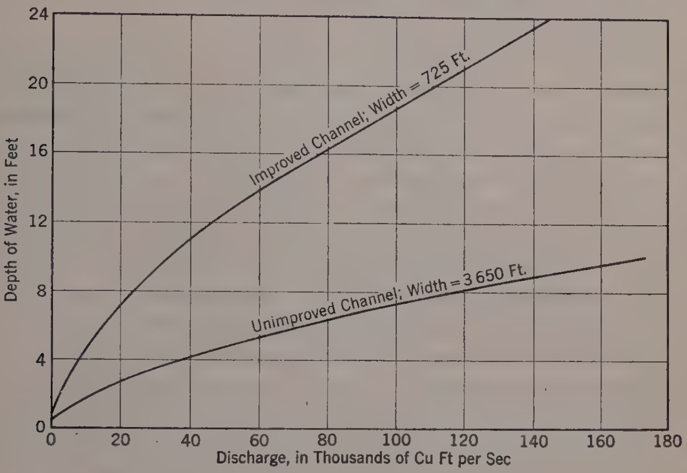


FIG. 11

The deepening of the channel will result in a cutting down of the bottoms of all tributary streams that are not restrained by rock ledges or deposits of coarse material. After the adjustment of channel cross section is substantially completed, at high discharges the water levels in the Missouri River will differ little from those at present; but at lower discharges in the contracted channel they will be considerably lower than in the unimproved conditions, and the bottoms of tributary streams will tend to cut down to lower levels, due to the increased fall that will be caused near their mouths as a result of the lowering of the bed in the main river. This action will bring into the main river an additional load of material that will have to be deposited in the unimproved channel area if the improved channel is to continue to deepen. This action will further slow up the deepening of the bed of the improved channel. There appears to be no way of making a close estimate of the rate at which lowering of the bed of the improved channel can be expected, except by using the experience in improved stretches as a guide.

The second phase of the effect of contraction, the gradual adjustment of the river gradient to the new conditions, will continue long after the adjustment of the channel shape to the confined conditions is practically complete. If the river were to decrease its slope from 0.96 to 0.76 ft per mile, as indicated by the computations of equilibrium slope, for the entire 760 miles of the length of the improvement, assuming that the elevation at the lower end remained the same, the upper end would be lowered 152 ft. Long before this occurred, the river would have cut below the bottom of the piles of the contraction works, and, unless they were replaced, the river would tend to resume its unconfined condition. The writer believes that what will happen is that the river will tend to flatten its slope, which will lower its bed and increase the inflow of sediment load from the tributaries due to the lowering of their beds resulting from the steeper slopes produced near their mouths, as previously explained. An increase of sediment inflow at the upper end of the improvement will also result from the lowering of the main river bed above this point. The flattening of the grade due to confinement will continue until the sediment inflow will increase and become equal to the capacity of the stream to transport material in its confined condition, and when this condition is reached no further decrease in slope will be observed. It is the writer's belief that this condition of substantial equilibrium will occur before much lowering of the stream bed due to changed slope has occurred, although it is probable that the lowering at the upper end will be sufficient to be easily detectable in the stage-discharge relations at that point. It is doubtful if the mean slope of the river will be changed more than 2% or 3% for many years to come, which substantially agrees with the author's statement that the change will not be appreciable.

If this be true, what then is the "equilibrium slope" computed by the author? This is the slope at which the confined river would carry the same load of material as the unconfined river carried at its natural slope. Soon after the works are constructed, however, the load to be carried will be increased and a slope greater than the computed equilibrium slope will be required.

The author states that the Schoklitsch formula could not be used to compute the equilibrium slope, which is true for the form given in Eq. 2. An examination of the paper, however, shows that this form was obtained by Schoklitsch by multiplying the load and discharge per unit width by the width, thus obtaining the equation in terms of the total load and total discharge. It can be expressed as well in terms of the load and discharge per unit width, which gives

$$G_1 = \frac{86.7\,S^{1.5}}{\sqrt{D}}\,(Q_1 - q_0)\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

in which G_1 and Q_1 are the sediment discharge per unit width and water discharge per unit width, respectively. Using this equation with the formula for q_0 given by Schoklitsch, it is possible to compute the author's equilibrium slope by this formula.

In discussing the formula developed by Professor Straub, the author states that the quantity of sediment must be assumed to be the same for the improved or unimproved channel. As previously shown, the quantities of sediment would not be the same in these two cases, and hence the formula will not give the slope that will actually be obtained in the river; it will give a flatter slope. The author also states that the sediment characteristics must be the same in the contracted as in the uncontracted stream. If the size of material increases with depth as shown in the analysis of samples given in Fig. 9, this condition also is not fullfilled, since coarser material would be encountered as the channel scoured deeper, and a steeper slope than computed would result from this cause.

There is considerable reason to question, however, whether accurate results can be obtained in computing the effect of contraction works on the Missouri River by considering only the bed load. The suspended load of sand carried by this river is very large, as shown by Fig. 9. A comparison of this observed sand load at Omaha and Yankton with the bed loads computed according to the Schoklitsch formula for a wide range of discharges, as shown in Tables 3 and 4, indicates that the bed load is probably never as much as 10% of the suspended load of sand in the Missouri River. Of course, it is a much smaller part of the total load, but the sand particles settle rapidly and therefore have a much larger effect on stream formation than the silt and clay sizes. The suspended sand load no doubt consists, on the average, of finer material than

TABLE 3.—COMPARISON OF BED LOAD AND SUSPENDED SAND; MISSOURI RIVER, AT OMAHA, NEBR.

(All Sediment Discharges Are in Thousands of Tons per Day)

SIZE OF SAND PARTICLES, IN MM		TEN-FOOT DEPTH			FIFTEEN-FOOT DEPTH			TWENTY-FOOT DEPTH			TWENTY-FIVE-FOOT DEPTH		
From	To	Bed load (1)	Suspended load (2)	(%) Col. 1 ÷ Col. 2 (3)	Bed load (1)	Suspended load (2)	(%) Col. 1 ÷ Col. 2 (3)	Bed load (1)	Suspended load (2)	(%) Col. 1 ÷ Col. 2 (3)	Bed load (1)	Suspended load (2)	(%) Col. 1 ÷ Col. 2 (3)
Discharge[a]			29,400			58,000			93,800			136,000	
Observed[b]		1,552	39,000	4.0	3,369	70,000	4.8	5,646	230,000	2.5	8,322	425,000	2.0
Computed[c]		....	25,350	84.5[c]	....	83,000	118.5[c]	....	197,000	85.7[c]	....	399,000	93.8[c]
0.05	0.075	33.3	7,750	0.43	67.4	25,350	0.27	110	76,100	0.14	160	161,500	0.10
0.075	0.10	83.1	6,600	1.26	169.5	16,750	1.01	278	41,100	0.68	405	71,600	0.57
0.1	0.15	224	6,440	3.48	465	21,300	2.18	767	41,800	1.83	1,120	75,300	1.49
0.15	0.25	765	4,120	18.5	1,639	15,950	10.3	2,740	35,700	7.69	4,020	77,800	5.17
0.25	0.30	306	453	67.5	678	1,880	36.0	1,145	4,920	23.3	1,690	10,650	15.85
0.30	0.40	98.2	40.0	245	227	188	121	390	605	64.4	581	1,468	39.60
0.40	0.60	38.4	2.0	1,870	99	0	∞	175	20	874	262	962	27.2
0.60	0.80	4.8	0	∞	15.1	0	∞	27.6	0	∞	42.8	0	∞
0.80	1.20	2.2	0	∞	10.8	0	∞	21.6	0	∞	33.7	0	∞

[a] Approximate discharges, in cubic feet per second. [b] Comparison of computed total bed load to observed total suspended load. [c] Comparison of computed total suspended load (Col. 3) to observed total suspended load (Col. 3).

the bed load, and some sizes carried by bed load are moved in suspension in negligible quantities, if at all. In Tables 3 and 4 are also given the computed bed loads for various size ranges and the quantities carried in suspension of these same size ranges computed by the method developed by A. A. Kalinske,[19] Assoc. M. Am. Soc. C. E., and the writer. This comparison shows that the suspended loads are greater than the bed loads for all sizes less than about 0.3 mm at Omaha and 0.4 mm at Yankton, and for high discharges, where the most active bed formation exists, for all sizes less than about 0.6 mm at Omaha and 0.5 mm at Yankton. In order to indicate the reliability of the relation used in computing the suspended load, a comparison is also given between the computed total load of sand and the observed load, as shown in Fig. 9. This comparison shows a reasonably close agreement of the computed with the mean of the observed loads. In view of the data shown in Table 3, it does not seem unreasonable to believe that in the Missouri River the suspended load may be as potent as the bed load in the channel-forming process.

TABLE 4.—COMPARISON OF BED LOAD AND SUSPENDED SAND; MISSOURI RIVER AT YANKTON, N. DAK.

(All Sediment Discharges Are in Thousands of Tons per Day)

SIZE OF SAND PARTICLES, IN MM		FIVE-FOOT DEPTH			EIGHT-FOOT DEPTH			TEN-FOOT DEPTH			TWELVE-FOOT DEPTH		
From	To	Bed load (1)	Suspended load (2)	(%) Col. 1 ÷ Col. 2 (3)	Bed load (1)	Suspended load (2)	(%) Col. 1 ÷ Col. 2 (3)	Bed load (1)	Suspended load (2)	(%) Col. 1 ÷ Col. 2 (3)	Bed load (1)	Suspended load (2)	(%) Col. 1 ÷ Col. 2 (3)
Discharge[a]			32,700			71,800			104,000			141,000	
Observed[b]		2,310	28,000	8.3	6,480	165,000	3.9	9,950	330,000	3.0	13,880	425,000	3.3
Computed[c]		....	47,600	170[c]	....	141,800	86[c]	....	268,000	81.2[c]	.....	431,000	101[c]
0.05	0.1	261	22,950	1.14	598	69,500	0.86	878	112,000	0.78	1,200	175,000	0.69
0.1	0.2	680	20,600	3.30	1,635	53,900	3.03	2,420	115,000	2.1	3,330	180,000	1.85
0.2	0.3	522	3,530	14.8	1,350	14,350	9.4	2,040	29,500	6.9	2,830	57,400	4.9
0.3	0.4	405	617	65.6	1,145	3,690	31.0	1,760	10,700	16.4	2,460	15,200	16.2
0.4	0.5	350	0	∞	1,105	388	285	1,730	1,410	123	2,450	3,200	76.6
0.5	0.7	119	0	∞	478	0	∞	775	281	275	1,080	380	284
0.7	1.0	13	0	∞	113	0	∞	195	0	∞	291	0	∞
1.0	2.0	−40	0	∞	55	0	∞	133	0	∞	222	0	∞

[a] Approximate discharges in cubic feet per second. [b] Comparison of computed total bed load to observed total suspended load. [c] Comparison of computed total suspended load (Col. 3) to observed total suspended load (Col. 3).

There seems to be little doubt that the confinement of the Missouri River will cause an increased flow of heavy sediment in it, and that this increased flow will be carried out into the Mississippi River. What will be the effect of thus adding to the load of that stream? Fortunately, the improvements previously built in the middle Mississippi River have been causing, over a long period, a lowering of the bed of this stream up to Chain of Rocks, a rock ledge across the river just below the mouth of the Missouri River. The extent of this lowering has become such that the water depths over the rock ledge have been greatly decreased and a by-pass canal with locks, or a movable dam with locks, is badly needed. The increase of the sediment load from the Missouri River, therefore, will tend to offset this lowering. Whether or not it will be sufficient to reverse the trend and cause a raising of the water levels downstream is a question; it is possible but unlikely. An increase in deposit, however, must occur at some point below the mouth of the Missouri River. Such effects are usually not considered in making river improvements, and their results usually do not become apparent for many years. Sometimes they require expensive corrective measures, as shown by the proposed improvements at the Chain of Rocks, previously cited.

---

[19] "The Relation of Suspended to Bed Material in Rivers," by E. W. Lane and A. A. Kalinske, *Transactions*, Am. Geophysical Union, 1939, Pt. IV, p. 637; also "Engineering Calculations of Suspended Sediment," *ibid.*, 1941.

ELLIOTT J. DENT,[20] M. AM. SOC. C. E. (by letter).[20a]—From Sioux City to Rulo the proposed channel will have a length of 245.6 miles, and in 1938 the total fall was 236 ft, or an average of 0.96 ft per mile. In his closing paragraph the author states that the improved channel will be able to carry 38% more bed load than the natural channel, or that it could carry the present bed load with a slope of only 0.76 ft per mile which corresponds to a drop of 186.5 ft in the reach under discussion.

If this section of the Missouri River is contracted and otherwise improved so as to increase its bed-load carrying capacity, it is logical to expect erosion at the upper end of the reach until a capacity load has been picked up. This increased bed load should then be carried through the remainder of the section and dumped upon the next lower reach at Rulo. To increase the bed load of the river channels materially, from Rulo to the Gulf of Mexico, is a matter that deserves very serious consideration.

---

[20] Col., U. S. Army (*Retired*), Washington, D. C.

[20a] Received by the Secretary October 20, 1941.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

LABORATORY INVESTIGATIONS OF SOILS AT FLUSHING MEADOW PARK

Discussion

BY DONALD M. BURMISTER, ASSOC. M. AM. SOC. C. E.

DONALD M. BURMISTER,[24] ASSOC. M. AM. SOC. C. E. (by letter).[24a]—It is very gratifying to find that Messrs. Thomas and Sinacori share many of the writer's views and have made practical use of correlation methods on a large project. In Figs. 9 and 10 they have shown how more effective methods were developed in borrow-pit studies, especially from the preliminary, and time and expense saving, standpoint, for obtaining information on the suitability of impervious rolled-fill materials for earth dams. The writer has made similar use of correlation methods in earth-dam investigations.[3] Although generalizations for all soil types are not possible, satisfactory regional correlations can be made for localized areas, where similar soil types are encountered; that is, soils formed essentially by the same geological processes and having essentially the same mineralogical character and grain-size distribution.

An important task of the engineer is to develop more effective methods of soil investigation for specific problems, especially from the time and expense saving standpoint. This involves the application of "Control Test Methods" by making a few carefully conducted and complete soil tests (consolidation, shearing, routine classification tests, etc.) on typical representative soils to define the correlation band or bands for the soil types encountered on the given project. The methods may be extended to the cohesionless soil types by using the "density limits," that is, the loose limit L, and the dense limit D, which serve the same purpose and reflect the physical character of cohesionless soils in a manner similar to the consistency limits for cohesive-plastic soils. Once a satisfactory correlation has been established, the simple routine tests can be made quickly and at a relatively small expense on a large number of

---

NOTE.—This paper by Donald M. Burmister, Assoc. M. Am. Soc. C. E., was published in January, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: May, 1941, by Gordon E. Thomas, Assoc. M. Am. Soc. C. E., and M. N. Sinacori, Jun. Am. Soc. C. E.; June, 1941, by E. J. Kilcawley, Esq.; September, 1941, by Dimitri P. Krynine, M. Am. Soc. C. E.; and October, 1941, by Messrs. B. K. Hough, Jr., and T. B. Rights.

[24] Asst. Prof., Civ. Eng., Columbia Univ., New York, N. Y.

[24a] Received by the Secretary October 14, 1941.

[3] "The Physical Characteristics of Soils with Special Reference to Earth Structures," by Donald M. Burmister, *Bulletin No. 6*, Dept. of Civ. Eng., Columbia Univ., New York, N. Y.

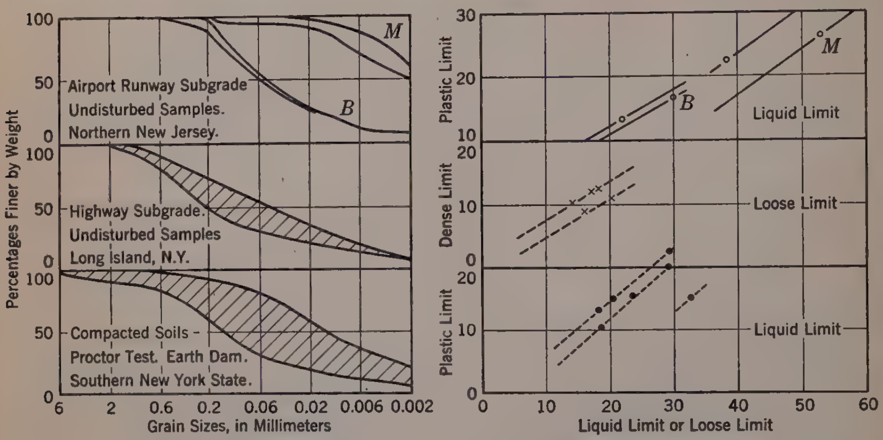
samples in order to obtain estimates of the consolidation, shearing, permeability, and other properties of soils, in as great a detail over an area as is considered necessary for an adequate investigation and solution of the problem. It is much more important to obtain information on the possible range of the estimates of these properties, which have a direct bearing on the analysis of the problem, as given by the width of the correlation band, than it is to know the exact values for a few isolated samples in a given area. Such an analysis would not only permit the engineer to visualize and analyze the situation more completely, but would also serve a very useful purpose for design and construction. Experience will be the best guide in teaching engineers how to make correct practical applications of this information.

To illustrate further the possibilities of such methods, data for three cases have been analyzed in Fig. 11 for the consolidation properties of the soils. This permits a more satisfactory classification either of artificially placed material or of natural deposits as to their relative settlement and bearing properties for highway subgrades, airport runways, and the like.

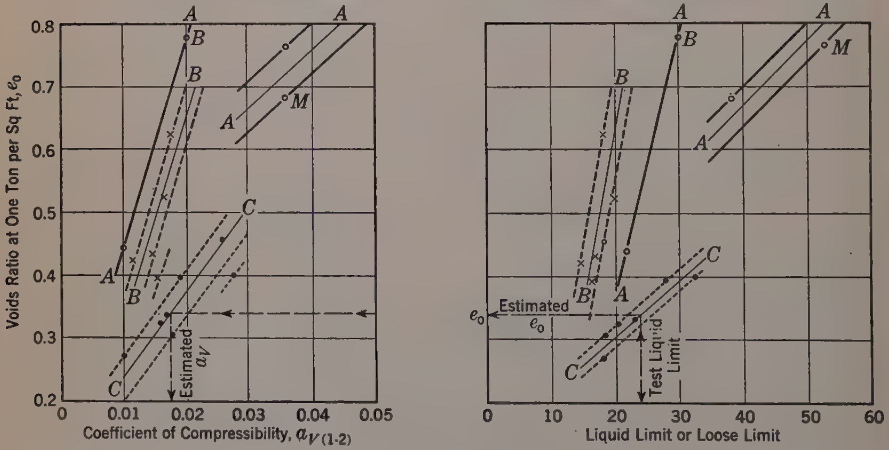
Professor Kilcawley states that an important part of foundation investigations is to present the field data as well as the results of laboratory tests so that subsurface conditions can be visualized clearly and interpreted correctly. This requires an accurate knowledge of the kinds of soils encountered—their extent, uniformity, and variability, their physical properties, and their general behavior characteristics. Consistency in the natural state is one of the most important of these simple characteristic properties. In very soft deposits, such as are encountered in the Flushing Meadow, the simple squeeze test is useful for obtaining fairly reliable information on the consistency and shearing strength of all soil samples. For medium stiff and hard clays, unconfined compression tests on small prisms can be made quickly on all samples for determining the shearing strength and to aid in selecting the least disturbed samples for more detailed analyses of their strength and stress-strain characteristics. Professor Kilcawley has emphasized the importance of establishing reliable guides for making estimates of the physical properties of soils and of establishing a maximum range of variation for soil type groups, as indicated by the width of the correlation band. One of the important problems in soil engineering is the accumulation of information on the behavior characteristics of the important soil type groups encountered in different sections of the United States, for example, such as those presented in Figs. 4, 9, 10, and 11, which will serve a useful purpose and help the engineer to analyze conditions for future construction, particularly for preliminary studies.

The observations by Professor Krynine on similar soft deposits in other localities are very instructive and provide additional information on their behavior. He has rightly stated that no civil engineering problem can be solved definitely in the laboratory. Mr. Hough, likewise, emphasizes the point that not only is it a matter of general interest to learn how test data are applied in practice, but the validity of the data cannot be judged apart from its application. Mr. Hough has suggested that further discussion on the application of data might well be undertaken.

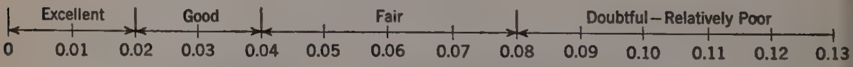
As an illustration, the "quick" direct shear tests and unconfined compression tests (see heading "Shearing Characteristics") provide preliminary information for estimating the safe height to which the initial fill could be placed in blanketing and reclaiming the site. Because of the impermeable character of the material, practically no increase in shearing strength could be counted



(*a*) GRAIN SIZE DISTRIBUTION (*b*) CONSISTENCY AND DENSITY LIMITS



(*c*) CORRELATION DIAGRAMS



(*d*) CLASSIFICATION OF SETTLEMENT AND BEARING PROPERTIES

FIG. 11.—CONSOLIDATION PROPERTIES

on over and above the natural shearing strength, as a result of consolidation. Such estimates should in every case be checked against field experience, or by a test fill as described by Mr. Freeman.[2] In the application of the theories of elasticity and plasticity to foundation problems, Mr. Jürgenson[11] has given the critical shearing stress conditions beneath the toe of an embankment approximately as follows: Plastic state—

---

[2] "The Application of Soil Mechanics in Building the New York World's Fair," by George L. Freeman, M. Am. Soc. C. E., Hamilton Gray, Jun. Am. Soc. C. E., and George W. Glick, *Civil Engineering*, October, 1940, p. 649; see also "Soil Survey of the Flushing Meadow Park Site, Long Island, New York," by George L. Freeman, *Paper No. C-2*, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Harvard Univ., Cambridge, Mass., 1936, Vol. I, p. 25.

$$S = \frac{p}{\pi} \qquad \text{(4a)}$$

and, failure—

$$S = \frac{p}{\pi + 2} \qquad \text{(4b)}$$

The shearing strength varies with depth as shown in Fig. 5. The situation would be more serious if it were not for the fact that the meadow mat had a very important spreading effect on stresses and strengthened the surface of the underlying soft material so that is was capable of supporting the initial blanketing to a height of about 4 ft and the grading equipment. The critical zone is deeper and is confined to a considerable extent by this mat layer; thus:

Depth	Plastic condition in meadow mat layer	Failure in underlying soft deposit
$p = w\,h =$	$2{,}000\,\pi\,S$	$2{,}000\,(\pi + 2)\,S$
h	$\frac{2{,}000}{100}\,\pi \times 0.08$	$\frac{2{,}000}{100}\,(\pi + 2)\,0.04$
h (in feet)	5.1	4.1

Experience in reclaiming and filling the Flushing Meadow showed that only in a few instances were any appreciable subsurface movements started, if the height of fill was kept within recommended safe limits.

To illustrate the application of the consolidation test results in a settlement analysis, the situation at boring No. 17, Figs. 1, 5, and 6, is considered. The following steps are taken in estimating the settlement to be expected as a result of a blanket fill, for example, of 10 ft:

1. The profiles of the overburden pressure, pre-consolidation pressure, and the pressure due to the fill, are estimated in Fig. 5 for boring No. 17, using the data on the approximate moisture content in Fig. 1 and the consolidation test data for two samples from boring No. 17 in Fig. 3 (also Fig. 2). The profile is subdivided, as shown, into a convenient number of increments of depth (located halfway between each sample, except at changes in the character of the soil).
2. The settlement property of the soils, which is defined by the compression index c_v, is estimated from the results of two consolidation tests in Fig. 3 and from the correlation diagrams of Fig. 4 for the soils of the Flushing Meadow for the remainder of the samples. The liquid limits of the samples of boring No. 17, which are the correlation arguments, are taken from Fig. 1. For settlement analyses it is more important to obtain an estimate of the possible range of the settlement properties of the soils in a deposit at the point where settlement estimates are desired than it is to know an exact value for an individual sample at a certain depth only. Accordingly, for the range of liquid limit for

---

[11] "The Application of Theories of Elasticity and Plasticity to Foundation Problems," by Leo Jürgenson, *Journal*, Boston Soc. of Civ. Engrs., July, 1934, p. 206.

these samples, the approximate maximum range of the estimates of the compression index c_v for the inner correlation band of Fig. 4 is found to be about ±15% from the mean curve of the correlation, which, it is believed, should be considered a fairly satisfactory accuracy considering all of the uncertainties involved in such analyses.

3. The natural stress conditions in the soil (which were discussed in the paragraphs preceding "Conclusions") are assumed for Case 1 to be represented by the dotted curve of Fig. 5, of pre-consolidation pressure (and complete consolidation). The approximate corresponding voids-ratio in the natural state (e_n) at the elevation of each sample is either taken directly from the pressure-voids ratio curves of Fig. 3 or is computed from Eq. 1. The stresses due to the 10-ft fill are computed on the basis of a unit weight (compacted) of 100 lb per cu ft.

4. In making the settlement estimates by the Increment Depth Method, Eq. 2 is solved by substituting the values of the voids ratios e_1 and e_2, corresponding to pressures p_1 and p_2 from Fig. 5 as follows:

$$\text{Settlement} = h \times \left(\frac{c_v}{1 + e_n}\right)\left(\log_e \frac{p_2}{p_1}\right) \ldots\ldots\ldots\ldots\ldots (5)$$

The approximate estimates of settlement are made as shown in Table 4, which contains the values obtained for each of the foregoing items.

TABLE 4.—APPROXIMATE SETTLEMENT ESTIMATES FOR BORING NO. 17, CASE 1, DUE TO A BLANKET FILL OF 10 FT

Sample No.	Depth, in ft	H, in ft	w_L	c_v	e_o	e_n	$\frac{c_v}{1 + e_n}$	p_1	p_2	$\log_e \frac{p_2}{p_1}$	S, in ft
..	0	6	(151.2)	0.815	3.52	4.77	0.141	0.10	0.60	1.38	1.17
1	6.0	5	151.2	0.517[a]	2.67[a]	3.39[a]	0.118[a]	0.23	0.73	1.154	0,69
..	11.0	..	....	....	....	....	....	....	....	....	....
2	17.0	11	104.5	0.500	2.45	3.04	0.123	0.25	0.75	1.095	1.47
3	27.5	10	105.3	0.506	2.45	3.10	0.123	0.27	0.77	1.045	1.29
4	38.0	11	116.2	0.680[a]	2.57[a]	3.41[a]	0.154[a]	0.29	0.79	1.000	1.70
5	49.5	14	106.0	0.511	2.48	3.06	0.126	0.32	0.82	0.940	1.65
..	57.0	..	....	....	....	....	....	....	....	....	....
6	59.0	7	(122.0)	0.635	2.90	3.30	0.146	0.50	1.00	0.460	0.47
..	64.0	..	....	....	....	....	....	....	....	....	....
											$S = 8.44$[b]

[a] Consolidation test values. [b] Possible range of estimates ±15%, or 7.3 to 9.7 ft.

To obtain some idea of the effect of differences between the possible stress conditions in the deposit and those assumed, the overburden pressure (full line) is assumed to represent the actual stress conditions for Case 2. The corresponding settlement estimate is equal to 7.7 ft. The corresponding settlement estimates for a 5-ft fill are equal to 5.3 and 4.9 ft, respectively. The settlement in the meadow mat layer at the top and the peat layer at the bottom probably occurred largely during the filling operation because of their more permeable character, as evidenced by the continuous flow of water from the toe of the advancing fill as it was constructed. Hence this settlement would be compensated for by additional fill in the operation of filling and grading to the required surface elevation, as suggested by Mr. Rights. Deducting these amounts, the estimates of subsequent settlement to be expected for a 10-ft fill (total height), for Cases 1 and 2, are 6.1 and 5.5 ft, respectively, and for a 5-ft fill, 3.9 and 2.7 ft, respectively.

The rate of settlement is estimated roughly on the assumption of an average value of the coefficient of consolidation, C_c, of 0.0012 cm² per min for the deposit, and of the natural voids ratio, e_n, of 3.32, and a deposit 47 ft thick, exclusive of the meadow mat and peat layers. The estimated times t in years, corresponding to different percentages of the ultimate settlement and time factors T_v, are estimated by the equation:[25]

$$t = \left(\frac{h}{1 + e_n}\right)^2 \frac{T_v}{1{,}400\, C_c} = 70.8\, T_v \,. \quad (6)$$

and are given in Table 5.

TABLE 5.—ESTIMATES OF THE RATE OF SETTLEMENT

Time[a]	PERCENTAGE OF SETTLEMENT				
	10	20	30	40	50
T_v.......	0.02	0.08	0.17	0.31	0.50
t........	1.4	5.6	12.0	22.0	35.4

[a] T_v = time factor; and t = time, in years.

In regard to the driving records of the casing and sampler, the writer agrees with Mr. Hough's statement that the number of blows per foot to drive the sampler indicates more reliably the resistance encountered at each elevation. In the Flushing Meadow deposits, only about one or two blows were required to drive the sampler because of its soft consistency. Since, for all important structures, pile foundations were required, the driving resistance of the casing was considered to be a better indication of total resistance, in this case.

In foundation studies for buildings and earth dams, the writer has devised and used the tentative values listed in Table 6, which has proved to be very useful for investigating and analyzing subsurface conditions with respect to their foundation properties.

TABLE 6.—TENTATIVE VALUES OF APPROXIMATE DEGREES OF COMPACTION

No.	Granular cohesionless soils	No. of blows[a]	Density[b] %	Cohesive plastic soils
1	Very loose silt..	<5	.....	Very soft clay
2	Loose sand or silt......................................	5–15	<40	Medium soft clay
3	Loose sand and gravel; medium compact sand or silt...	15–35	40–70	Medium stiff clay
4	Medium compact sand and gravel; compact sand or silt.	35–70	70–90	Stiff or medium hard clay
5	Compact sand and gravel; very compact sand or silt...	>70	90+	Very hard clay

[a] Blows per foot; 3-in. sampler; driving energy, 7,200 in-lb; weight of hammer, 300 lb; average drop, 24 in.
[b] Relative density, expressed as a percentage.

The liquid limit of cohesive plastic soils and the loose limit of cohesionless soils, as laboratory references, are assumed to be 40% relative density based on experience with the measured loosest states found in natural deposits. The plastic limit and the dense limit, respectively, as laboratory references, are assumed to be 100% relative density. On this basis very few natural deposits have a relative density greater than 100%.

---

[25] Progress Report of Special Committee on Earths and Foundations, *Proceedings*, Am. Soc. C. E., May, 1933, p. 777.

The relative densities D_d are determined by the relations: Cohesive plastic soils,

$$D_d = 40 + 60 \frac{w_L - w_N}{w_L - w_p} \qquad (7a)$$

and cohesionless soil,

$$D_d = 40 + 60 \frac{w_l - w_N}{w_l - w_d} \qquad (7b)$$

in which:[26] w_L = liquid limit; w_p = plastic limit; w_N = approximate natural moisture content; w_l = loose limit, as a moisture content; and w_d = dense limit, as a moisture content.

A distinction is made in the driving resistance of sand-gravel moistures and sand or silt because the presence of gravel increases the resistance quite as much as an increase in the degree of compactness. There is no simple direct relation for different sizes of samplers and casings, and therefore the numerical values must be determined from field experience. However, for different weights of hammers and drops, fairly consistent values are obtained, if the blows per foot are made inversely proportional to the driving energy (weight of hammer times drop).

The tentative values in Table 6 have not only made it possible to visualize subsurface conditions better as to their foundation properties, bearing capacity, length of piles, etc., but also have served a useful purpose in providing numerical values by which to compare different situations, particularly in cases where settlements are known, where loading tests have been made, etc.; or, in cases where the allowable bearing capacity is to be estimated—for example, from the Boston Building Code, which differentiates between the values for loose and compact conditions of the soils.

In conclusion, the writer wishes to express his gratitute to the discussers for their stimulating comments.

---

[26] See "Soil Mechanics Nomenclature," *Manual of Engineering Practice No. 22*, Am. Soc. C. E.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

CONCRETE IN SEA WATER: A REVISED VIEWPOINT NEEDED

Discussion

BY T. B. RIGHTS, ASSOC. M. AM. SOC. C. E.

T. B. RIGHTS,[33] ASSOC. M. AM. SOC. C. E. (by letter).[33a]—Experience has led the writer to believe that concrete should never be used in sea water between high-tide and low-tide elevations. If there is disintegration in concrete in sea water, above the high-tide level, the factor of the brand of cement used in the construction should be investigated carefully.

In 1925, the engineers of an eastern railroad built a bridge across an estuary near New York, N. Y. At the eastern end of this bridge, two test piers were constructed. Pedestal piers that supported the steel work of the bridge protected these test piers, so that mechanical abrasion due to floating ice or logs was reduced to a minimum.

The test piers were identical except for the cement. The bottoms were about 2 ft below mean low water and the tops 2 ft above mean high water. No reinforcement was placed in either pier. Fresh water was used for mixing water. Both test piers were constructed "in the dry" with accurately measured water, and proper batching of aggregates. One test pier was built with the same brand of cement as the remainder of the bridge; the other was constructed with quick-setting high-strength cement.

Within two years after construction, the test pier made of the quick-setting cement began to disintegrate and within five years had disintegrated so much that it could only be seen at low water.

Twelve years after construction, the test pier of regular cement was sound, and apparently unaffected by years of exposure to the tidal range of sea water. At that time, one would draw the conclusion that the failure of the one test pier was due only to the cement. At present (only sixteen years after construction), the regular cement test pier is beginning to disintegrate.

NOTE.—This paper by Homer M. Hadley, Assoc. M. Am. Soc. C. E., was published in January, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1941, by Thomas E. Stanton, M. Am. Soc. C. E.; April, 1941, by Messrs. W. F. Way, and Glenn S. Paxson; May, 1941, by Messrs. Lester C. Hammond, Ladis H. Csanyi, and G. M. Williams; June, 1941, by Messrs. Harry E. Squire, and J. W. B. Blackman; and October, 1941, by Messrs. E. C. Jack, and Alfred M. Freudenthal.

[33] Roselle, N. J.

[33a] Received by the Secretary October 16, 1941.

Engineers of this railroad company find that, as a general rule, concrete of the better grade will not show disintegration for at least twelve years. Consequently, on important bridge piers built in salt water, it is their general practice to face the piers between a plane about 2 ft below low water to a height of 2 ft above high water, with either granite or hard paving blocks. The railroad bridge cited herein, where the test piers are situated, has its many piers constructed in this manner, with a belt of granite blocks between extreme high and low water. There is no apparent disintegration in any piers thus protected. The concrete above the granite blocks seems (1941) to be in excellent condition. However, the mortar between the granite blocks, on the horizontal layers, is missing in a number of these piers. In fact, at some piers it seems as if only the bond at the backs of the granite blocks holds them in place. This railroad company has not experienced trouble with any of its concrete piers poured in fresh water. The only zone of attack and disintegration appears to be in sea water between the tidal range.

Figs. 2 and 4 apparently show disintegration at the base of the wall. If a belt of granite had been used, the structure might be in good condition today, as the remainder of both walls above high tide appears sound.

The writer does not know what caused this disintegration—whether it is sulfate-of-magnesium attack, freezing, or crystallization of the salt in the water—but he does know that if he were to construct piers in salt water, the concrete would be protected by a belt of granite at the tidal zone.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852

DISCUSSIONS

* TUNNEL CONSTRUCTION, SIXTH AVENUE SUBWAY, NEW YORK, N. Y.

Discussion

BY JOHN H. MYERS, M. AM. SOC. C. E.

JOHN H. MYERS,[5] M. AM. SOC. C. E.[5a]—There was a time when "rock men" had in mind only progress. Inasmuch as their work was done largely in the wilderness, it was not necessary to give much attention to safety. The rock men of that day were adepts in blowing down mountains and incidentally scattering spalls all over the landscape, or in making, with the machinery then available, what was considered remarkable progress in tunnel excavation. However, when engineers began to build large structures within existing cities, progress, tempered with the necessary precautions for the safety of life and of existing structures, was inevitable and as a result a new breed of rock men has been developed. Mr. Feld gives much evidence of this fact and he has done something very worth while in recording the methods used in the construction of the Sixth Avenue Subway.

The so-called "umbrella section" of double-track tunnel, described by Mr. Feld, has been used for many years in the tunnel work of the New York Rapid Transit Railroads. To the writer's knowledge, it was used in many places on the New York City subway work, among them being the connection between the Lexington Avenue Subway and the existing tunnels under Park Avenue; and also at many locations along Lexington Avenue north of the point where the subway emerges from under the Commodore Hotel. In fact, some of the tunnels under Lexington Avenue are two tracks high. The upper half of the tunnel is of the "umbrella section" and rests on a steel floor which is at the same time the roof of the lower half.[6] The fact that the "umbrella section" is still being used gives proof that although, when completed, it furnishes an adequate encasing structure for the Rapid Transit Railroad tracks, it is, at the same time, a practical structure to build under many difficult conditions.

NOTE.—This paper by Jacob Feld, M. Am. Soc. C. E., was published in April, 1941, *Proceedings*.

[5] Div. Engr., Board of Transportation (*Retired*), White Plains, N. Y.

[5a] Received by the Secretary October 16, 1941.

[6] "Construction Problems of the Manhattan-Bronx, and Lexington Avenue Subway Junction and Queensborough Tunnel Connections," by George Perrine, *Transactions*, Am. Soc. C. E., Vol. LXXXII, December, 1918, p. 278; see also "Tunnel Work on Sections 8, 9, 10, and 11, Broadway-Lexington Avenue Subway, New York City," by Israel V. Werbin, *ibid.*, Vol. LXXXI, December, 1917, p. 341.

In the writer's experience with it, the difficult conditions were brought about by the rock encountered, and he has known the "umbrella section" to be constructed by many different methods. At times it was built by excavating a central drift, placing the concrete center bench on the floor of this drift, erecting the permanent steel work, bracing the rock roof from this steel work and then concreting the "umbrella." Following this step, the tunnel for one track was excavated without further timbering, the arch over the track was concreted, and temporary bracing was placed against the unexcavated rock on the other side of the center line to take up the unbalanced thrust of the completed half tunnel. The remaining half was then excavated and the concrete poured. In Mr. Feld's experience, however, he had to cope not only with bad rock conditions but, in addition, with buildings alongside the tunnel work and subsurface structures, and an elevated railroad above it. All in all, it serves to demonstrate that the "umbrella" form of tunnel is a flexible design and that it is possible to erect it under many and various difficult conditions. The designers of the subway construction have no reason to be ashamed of it.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852

DISCUSSIONS

COST OF PUBLIC SERVICES IN RESIDENTIAL AREAS

Discussion

BY HOWARD WHIPPLE GREEN, M. AM. SOC. C. E.

HOWARD WHIPPLE GREEN,[10] M. AM. SOC. C. E.[10a]—With the United States approaching a stable population and its large cities beginning to realize that their population increases will be much more moderate in future decades, with several large cities for the first time showing actual decreases in population while their suburbs still continued to increase, with fewer and fewer persons left in the central city to bear larger and larger tax burdens, with industry and commerce showing indications of decentralization, and with slum and blighted areas increasing in extent and severity, the problem of rebuilding cities in the United States ranks near the top of the list of important things to be given attention following the present national emergency.

A possible solution to this serious problem of the depopulation of the center of most large American cities with increases in population around the periphery is indicated by the detailed calculations presented in this paper. Although all computations presented are naturally based upon New York City, standards of land use, and unit costs of public improvements and services, similar computations made in smaller cities with different standards of land use and different unit costs may demonstrate even more striking differences between the rehabilitation of existing blighted areas, the entire rebuilding of these areas, and the relocation of their populations on vacant land in outlying sections within or beyond their corporation limits.

The basic facts made available in Cleveland, Ohio, by the continuous Real Property Inventory started in October, 1932, augmented by unit costs of public improvements and services, make such a study possible with a minimum amount of effort.

Cleveland being built horizontally, contrasted with New York's vertical pattern (especially the Borough of Manhattan), doubtless will show quite

---

NOTE.—This paper by F. Dodd McHugh, Esq., was published in June, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: October, 1941, by Donald M. Baker, M. Am. Soc. C. E.

[10] Secy., Cleveland Health Council; Director, Real Property Inventory of Met. Cleveland, Cleveland, Ohio.

[10a] Received by the Secretary October 14, 1941.

different factors. However, it is as true in Cleveland as in New York that the public utilities that were complete in the section designated as census tract J-1 (with a population of 6,114 in 1910), which decreased to 493 in 1920, to 49 in 1930, and to 3 in 1940, were of no assistance to census tract T-8 with a population of 59 in 1910, which increased to 1,557 in 1920 and to 10,577 in 1930, decreasing to 9,986 in 1940; nor were the schools, hospitals, and churches, or the retail stores, water mains, sewers, paving, gas lines, and electric and telephone networks serving the 6,114 inhabitants of census tract J-1 in 1910 and quite useless to the 3 persons living in this area in 1940 of any aid to the suburb of University Heights, which increased from a population of 156 in 1910 to 5,981 in 1940. Such a shifting of population is one problem during periods of rapid population expansion accompanied by proportional expansions of industry and commerce, but it is quite another as a stable population is approached.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

A DIRECT METHOD OF FLOOD ROUTING

Discussion

By Messrs. W. B. Langbein, Fred E. Tatum, and H. C. Woster

W. B. Langbein,[20] Assoc. M. Am. Soc. C. E.[20a]—A reading of this paper seems to give the impression that the objectives of flood routing as set forth in the first paragraph are not being realized at present. In their presentation the authors seem to overlook the fact that floods are being routed daily and expeditiously by empirical methods as described in the following references which are evidently as simple as those they proposed:

(1) "Flood of Ohio and Mississippi Rivers, January–February, 1937," by N. C. Grover, M. Am. Soc. C. E., *Water-Supply Paper No. 838*, U. S. Geological Survey, 1938.
(2) "Natural Stream-Channel Storage," by R. E. Horton, M. Am. Soc. C. E., *Transactions*, Am. Geophysical Union, 1936, pp. 406–415.
(3) "Definitions and Classification of Flood Waves," by R. E. Horton, *Bulletin No. 25*, Permanent International Assoc. of Navigation Congresses, Brussels, Belgium, 1938.
(4) "A Method for Correcting River Discharge for a Changing Stage," by B. E. Jones, M. Am. Soc. C. E., *Water-Supply Paper No. 375*, U. S. Geological Survey, 1916, pp. 117–130.
(5) "Some Channel-Storage and Unit-Hydrograph Studies," by W. B. Langbein, *Transactions*, Am. Geophysical Union, 1940, pp. 620–627.
(6) "The Unit Hydrograph and Flood Routing," by G. T. McCarthy (unpublished manuscript presented at conference of North Atlantic Div., Corps of Engrs., U. S. Army, June 24, 1938).
(7) "Hurricane Floods of September 1938," by C. G. Paulsen, M. Am. Soc. C. E., and others, *Water-Supply Paper No. 867*, U. S. Geological Survey, 1940.
(8) "Flood Routing," by E. J. Rutter and Q. B. Graves, Assoc. Members, Am. Soc. C. E., and F. F. Snyder, Jun. Am. Soc. C. E., *Transactions*, Am. Soc. C. E., Vol. 104 (1939), pp. 275–313.
(9) "River Hydraulics," by the late J. A. Seddon, M. Am. Soc. C. E., *ibid.*, Vol. XLIII, June, 1900, pp. 179–243.
(10) "Engineering Construction: Flood Control," the Engineer School, U. S. Army, Fort Belvoir, Va., 1940, pp. 127–177.

Note.—This paper by C. O. Wisler, M. Am. Soc. C. E., and E. F. Brater, Jun. Am. Soc. C. E., was published in June, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: October, 1941, by Messrs. Ray K. Linsley, Jr., Harold C. Hickman, Robert B. Horonjeff and Herbert G. Crowle, L. K. Sherman, and Alfred L. Brosio.

[20] Associate Hydr. Engr., U. S. Geological Survey, Washington, D. C.

[20a] Received by the Secretary October 15, 1941.

(11) "Synthetic Unit-Graphs," by F. F. Snyder, *Transactions*, Am. Geophysical Union, 1938, pp. 447–454.

(12) "The Hydraulics of Flood Movements in Rivers," by H. A. Thomas, M. Am. Soc. C. E., *Engineering Bulletin*, Carnegie Inst. of Technology, 1934.

(13) "Storage and the Unit Hydrograph," by C. O. Clark, Jun. Am. Soc. C. E., U. S. Engr. Office, Norfolk, Va., 1940 (unpublished).

The authors undertake to compute a hydrograph of inflow from the unmeasured area and refer (paragraphs preceding "Routing Q and I Through to Morgantown") to a comparison between two methods of computing the volume of inflow. It appears to the writer that the two methods are one; any difference should be ascribed to arithmetic discrepancies.

For example, in method 1, $\sum$ outflow at Morgantown $-\sum$ flow at Fetterman = total intervening inflow, whereas in method 2,

$$I = \frac{O_2 + O_1}{2} - \frac{Q_2 + Q_1}{2} + \frac{S_1 - S_2}{\Delta t} \qquad (14)$$

in which Δt is time intervening between events indicated by subscripts 1 and 2.

From the beginning of the hydrograph to its end, storage fluctuates from zero back to zero; hence,

$$\sum I = \sum \frac{O_2 + O_1}{2} - \sum \frac{Q_2 + Q_1}{2} \qquad (15)$$

as before. Hydrographs of channel inflow from intervening, partial, and total basin areas were computed by the U. S. Geological Survey by Eqs. 14 and 15 as a part of its analysis of the Ohio River flood of January, 1937 (1).[21]

The authors state (see "Synopsis") that

> "The entire procedure is based upon the storage equation and upon the principal that, for all high stages, there is a straight-line relationship between the volume of storage contained in any reach of river channel and the sum of the inflow rate at the upper end and the outflow rate at the lower end of that reach. Except perhaps for unusual channel conditions, this relationship holds true."

This statement is not generally supported by studies of others. G. T. McCarthy, Assoc. M. Am. Soc. C. E. (6), showed that, in the general case, unless proper weights were assigned to inflow and outflow, the discharge-storage relation was a loop. The special case of the authors, of equal weights to outflow and inflow, has been shown by the writer (5) and others (13) to be comparable to simple translation.

The straight-line relationship between weighted discharge and storage is approximated by many reaches. It is a convenience but not a necessary corollary to general flood-routing technique, and is not a proof of the authors' general statement that overland flow ceases and channel storage-drainage begins at the time of the point of contraflexure on the recession side of the hydrograph. It may be true for a minor tributary but certainly does not hold true for larger basins. For example, in Fig. 1 the point of contraflexure occurs later on the

---

[21] Numerals in parentheses, thus: (1), refer to the foregoing references to current literature on "Channel Storage and Flood Routing."

Morgantown hydrograph than on the one at Fetterman; evidently, this is simple hydraulic lag. The time of contraflexure was even later at Charleroi, Pa., and, as one proceeds downstream, even the peak may be considered as drainage from upstream areas from which overland flow had long since terminated.

The writer has used the method of computing storage relations from recession curves (5). From his experience he would be very much surprised if points prior to the point of contraflexure did not plot to the right of the curve as in Fig. 2, and he cannot accept this as proof that this is the point of termination of overland flow.

The method of verification adopted by the authors (applying empirical formulas directly to the material used to derive them) may be deficient to the extent that the data used in the derivation of the technique may have sampled only a limited number of experiences. A later flood representing an entirely different combination of events may prove such empirical methods to be unreliable. Present flood-routing technique has many shortcomings, and there is a great need for the application of some of the more general laws of fluid motion to the problem to replace present empiricisms. It seems probable that the authors did not take full advantage of the considerable amount of published material relating to the technique of flood routing.

FRED E. TATUM,[22] JUN. AM. SOC. C. E.[22a]—The method advanced by the authors for obtaining their storage curves saves considerable time. The curve thus obtained represents the true storage for that particular storm rather than the average curve as computed by other methods.

The true recession curve is an exponential curve[23,24] and the recession factor can be designated by "K." The recession factor "K," of both the upper and lower gaging stations, must be equal for the S versus $(Q + O)$-curve to be a straight line. In the case of unequal recession factors, S versus $(Q + O)$ will plot as a curved line.

Since the recession curves are exponential and equal, the S versus $(Q + O)$-curve should be a straight line starting at the origin and drawn parallel to the curve used by the authors. The curved line obtained at lower stages by the authors is apparently due to the fact that the point at which surface runoff appeared to end was sooner than the actual ending time.

This routing method gives good results when the two gages are reasonably close together, but if the reaches are too long for the period of time required for routing, the results cannot be depended upon. One of the requirements in routing is that the reach should be of such length that the time of travel through the reach is about the same as the length of the period used in routing.

It seems that the authors went to considerable trouble in computing the local inflow and then routing it down to the gage, when it is readily obtainable by an easier method. The hydrograph of local inflow, of necessity, would be at the lower gaging station, and therefore would not need to be routed. The local inflow hydrograph can then be obtained by subtracting the routed flows of the upper gaging station from the flows of the lower gaging station. The difference will be the hydrograph for the local inflow.

---

[22] Junior Engr., U. S. Engrs., Rock Island, Ill.

[22a] Received by the Secretary October 15, 1941.

[23] "Relation Between Rainfall and Run-Off from Small Urban Areas," by W. W. Horner and F. L. Flynt, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 140.

[24] "The Mathematical Synthesis of the Flood Hydrograph," by R. T. Zoch, *Transactions*, Am. Geophysical Union, 1939, p. 207.

H. C. WOSTER,[25] ASSOC. M. AM. SOC. C. E.[25a]—The method of flood routing developed by the authors is felt to be a distinct contribution to the methodology of delineating flood and damage source areas. The method is particularly adapted to use in the flood-control preliminary examinations and surveys of the Department of Agriculture, in the prosecution of which it is usually impracticable and economically impossible to obtain the channel cross sections and other hydraulic data requisite to the more exact methods of flood routing.

The authors' method has been used during and since its development in the flood-control preliminary examinations conducted by the Appalachian Forest Experiment Station, at Asheville, N. C., and, since May, 1941, under the supervision of the Regional Office, Bureau of Agricultural Economics, at Atlanta, Ga. The method was used first as a means of defining the principal sources of flow contributing to downstream damages, but in studies made since May, 1941, routings made by this method have been used as a basis for allocating actual flood damages at critical downstream points to the respective tributary areas. The allocation of damages to source areas in turn becomes one of the bases for determining the critical areas on which a land-use improvement program can be expected to contribute to flood reduction. The flow routing is used also for approximating the contribution of each area to the anticipated reduction.

This discussion is intended to present some of the difficulties encountered in the application of the method, in the hope that they may be considered by the authors in any further development and refinement of the method. It is regretted that solutions of the difficulties cannot be offered in this discussion.

The authors state that their selection of gaging stations for illustration was based partly on the excellent discharge records available. It is evident from the presentation that very good results have been obtained with these records. In the studies made by the writer it has been necessary to use the available discharge records for the watersheds studied, many of which are less reliable than those used by the authors because of lack of definition of stage-discharge relationships for high stages or because of complications due to backwater, slight regulation, or other causes. Gaging station records affected by pronounced stream regulation have not been used. The results obtained by using the regular run of discharge records are less fortunate than those encountered by the authors. It has been found necessary in most cases to adjust the routed hydrographs of Q and I after completion of the procedure because when they are combined they do not check, even closely, the actual hydrograph. In some cases the adjustments have been rather large, necessitating the exercise of personal judgment and hence detracting from the usefulness of the routing method as an exact procedure.

---

[25] Associate Hydr. Engr., Appalachian Forest Experiment Station, U. S. Forest Service, Atlanta, Ga.

[25a] Received by the Secretary October 16, 1941.

In most cases where adjustment of the routed hydrographs has been made, it has been found necessary to reduce the peak of the routed hydrograph of Q and move it ahead in point of time. This appears to indicate that the relationship of $(Q + O)$ to channel storage does not yield sufficiently large storage values. This has been found to be particularly true in cases where the distance between gaging stations is long. One reason for this may be that the relationship between $(Q + O)$ and channel storage was developed from the recession side of the hydrograph and does not take account of the fact that for a given stage the channel storage is greater on the rising than on the recession side of the hydrograph. Consequently, the application of the authors' channel storage relationship reduces the peak discharge in transit between stations less than actually occurs. Also, the actual time of transit of the flood wave is greater than that obtained by this method.

It has been found that, in computing outflow from the inflow hydrograph of Q, negative values of outflow are often obtained for a number of periods at the beginning of the outflow hydrograph. Since the total volume of Q', including the negative values, equals the original volume of Q, then, if the negative values of Q' are called zero, the total outflow becomes greater than the original volume. This is believed to be due to errors in the slope of the storage curve near its lower end and doubtless could be corrected by adjusting the slope of the storage curve by trial and error. It has been found to be quicker and more convenient, however, to adjust the hydrograph of Q' directly to the original volume.

Correction for *Transactions:* In the inset of Fig. 1, the stream labeled "Monongahela River" is the "West Fork River."

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

METHOD OF PREDICTING THE RUNOFF FROM RAINFALL

Discussion

By Messrs. Richard van Vliet, and LeRoy K. Sherman

Richard van Vliet,[8] Esq.[8a]—Some of the factors that enter into the so-called water-loss curve have been evaluated in this paper. Water-loss curves based on monthly or seasonal rainfall have been used for a number of years by hydrologists in studying the yield of drainage basins. The authors have developed such a curve for the Valley River, based on individual stream rises. The writer wishes to comment on a few of their conclusions.

The authors refer to one of the ways by which rainfall is disposed of as "surface loss" which, as defined by the authors, is in reality the sum of two quantities—interception and depression storage.[9] In order to determine these quantities separately, a study of the storm rainfall pattern is required. It cannot be done with any degree of success by considering only the total amount of storm rainfall and duration. The volume of interception and depression storage depends on rain intensity. Interception occurs at all rainfall intensities, whereas depression storage cannot occur unless the rate of rainfall is greater than the infiltration capacity. The authors state that "surface loss" becomes constant at some extremely high rainfall, but this is true only if the particular "high rainfall" is equal to the infiltration capacity. Excluding infiltration, interception becomes constant at very moderate rain intensities after interception storage is filled. This paper is a study of surface runoff phenomena, and in such a study the rainfall must be considered in detail, in regard to both its intensity and duration.

The authors state that the duration of surface runoff may be taken as the length of the unit graph base. From analyses of various stream rises, together with the corresponding channel inflow graphs, it has been found that direct surface runoff ends at about the point of inflection on the recession side of the

---

Note.—This paper by Ray K. Linsley, Jr., and William C. Ackermann, Juniors, Am. Soc. C. E., was published in June, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: October, 1941, by Bertram S. Barnes, Assoc. M. Am. Soc. C. E.

[8] Prin. Asst. Engr. to Robert E. Horton, Cons. Hydr. Engr., Voorheesville, N. Y.

[8a] Received by the Secretary October 8, 1941.

[9] "Hydrologic Interrelations of Water and Soils," by Robert E. Horton, *Proceedings*, Soil Science Soc. of America, Vol. 1, 1937, pp. 401–429.

hydrograph of a stream rise.[10] Beyond the point of inflection the recession side of the hydrograph is composed of two items—channel storage and ground water. The area under the recession side of the hydrograph beyond the point of inflection and above the ground-water flow curve is usually referred to as "channel storage." The use of the term "surface-runoff recession curve" by the authors to express this quantity is deemed unfortunate, as the term "channel storage recession curve" is generally used to express this quantity, since surface runoff has ended. Attention is called to the fact that the method of developing channel storage recession curves was first published by Robert E. Horton,[10] M. Am. Soc. C. E.

In Fig. 7 the authors correlate "duration of rainfall, in hours," with "time to ground-water peak, in hours." A study of the paper leads to an understanding that duration of rainfall is the total duration of storm rainfall, including any residual rainfall (rainfall in which the intensity is less than the infiltration capacity). The writer cannot agree with the authors that this relation holds true for all cases if total rainfall is accepted under their definition. For example, consider two storms, each having identical periods of rainfall excess of 1-hr duration; that is, with rainfall intensity greater than infiltration capacity. In one storm, however, assume also a period of residual rainfall in which the intensity is, say, 0.05 in. per hr, or well below the infiltration-capacity rate, and lasting for 4 hr. In the first storm the duration of rainfall is 1 hr and in the latter storm 5 hr. Other factors being equal, the time at which the ground-water flow will reach its peak will be practically the same for both storms. It seems to the writer that any attempt to correlate total duration of rainfall with time for ground water to reach its peak will give results of little value if the total duration of rainfall includes the period of residual rainfall.

LEROY K. SHERMAN,[11] M. AM. SOC. C. E.[11a]—The quantity of moisture in the soil materially governs the intake of rainfall and consequently affects the quantity of surface runoff. Several empirical procedures have been used to approximate this important antecedent soil-moisture index. The writer[12] and others have based their approximations on percentage of antecedent rainfalls together with the intervening amounts of "drying out" time. The authors have estimated the amount of "drying out" with the logical and more specific use of an evaporation factor.

In Table 1 the usual practice of segregating base flow, or ground-water flow, and surface runoff has not been followed. Apparently the authors assume that ground-water runoff (Col. 3) is due to infiltration, not absorbed (Col. 10), and that this ground-water runoff all comes from the given storm on the basin. This may or may not be true. Ground-water runoff may originate, in part, from territory outside of the basin. A part of the ground-water flow may be measured by the stream gage and a part of this flow may by-pass the gage in transit through the soil. It is possible to check the assumption made in this case.

---

[10] "Surface Runoff Phenomena," by Robert E. Horton, *Publication 101*, Horton Hydrological Lab., Voorheesville, N. Y., 1935.

[11] Cons. Engr., Chicago, Ill.

[11a] Received by the Secretary October 15, 1941.

[12] "The Unit Hydrograph and Its Application," by L. K. Sherman, *Journal*, Associated State Eng. Societies, April, 1941, p. 4.

In Table 3 the writer has selected, at random, eight storms from Table 1. If the ground-water runoff is all accounted for in the manner assumed by the authors, then the figures in Col. B will equal the corresponding figures in Col. A. The column of differences A − B is the error in the authors' assumption, predicated upon correctness of computed absorption and surface loss. For a pioneering job this showing is not bad. It indicates that in this particular basin the authors' assumption as to ground-water flow and origin is correct. The authors have not made use of the more recent concepts of the infiltration theory, such as rainfall excess, the recession curve, and the mass curve of infiltration capacity. The writer used a few of the authors' antecedent soil moisture figures in such tests, but the number so used was not sufficient to draw any conclusions.

TABLE 3.—CHECK COMPUTATIONS OF LOSS AND RUNOFF[a]

Date	Rainfall −	Surface runoff =	Infiltration and initial loss	Ground-water runoff +	Absorption +	Surface loss =	Infiltration and initial loss	Difference (Col. A minus Col. B)
	(1)	(2)	(A)	(3)	(10)	(11)	(B)	
2-13-35	1.19	0.14	1.05	0.42	0.20	0.34	0.96	−0.09
2-14-35	1.66	0.37	1.29	0.53	0.23	0.42	1.18	−0.11
3- 5-35	1.19	0.09	1.10	0.58	0.25	0.34	1.17	+0.07
3- 6-35	1.01	0.21	0.81	0.45	0.30	0.31	1.06	+0.25
3-10-35	0.77	0.03	0.74	0.19	0.40	0.27	0.86	+0.12
3-12-35	2.41	0.94	1.47	0	0.73	0.54	1.29	−0.20
4- 1-36	4.30	1.71	2.59	1.24	0.58	0.81	2.63	+0.04
4- 5-36	3.88	1.42	2.46	1.38	0.34	0.75	2.47	+0.01

[a] The column numbers correspond with those in Table 1.

Further research, on the derivation of initial soil moisture along the lines suggested by the authors, is much needed in applied hydrology.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

LAND SURVEYS AND TITLES

SECOND PROGRESS REPORT OF THE JOINT COMMITTEE OF THE REAL PROPERTY DIVISION, AMERICAN BAR ASSOCIATION AND THE SURVEYING AND MAPPING DIVISION, AMERICAN SOCIETY OF CIVIL ENGINEERS

Discussion

BY T. J. RODHOUSE, JR., ASSOC. M. AM. SOC. C. E.

T. J. RODHOUSE, JR.,[9] ASSOC. M. AM. SOC. C. E.[9a]—The recommendation made in Section II of the Joint Committee's Progress Report that appropriate measures be taken to hasten official recognition and use of the state plane coordinate systems by enlisting the cooperation of state agencies is certainly a step in the right direction. If state highway departments, and other departments of the various state governments whose use of surveys is substantial, will use the state coordinate systems and make available the coordinates of monumented points, legal recognition will undoubtedly be materially speeded up in most states.

It is notable that a state highway engineer was largely responsible for the start of the computations of the state systems of plane coordinates. Early in 1933, the late George F. Syme, M. Am. Soc. C. E., of the North Carolina State Highway and Public Works Commission, requested the U. S. Coast and Geodetic Survey to study the possibility of setting up systems of plane coordinates for North Carolina and thereby furnished the incentive for the initiation of such schemes.[10]

Certain agencies of the federal government are also in a position to aid materially in the campaign for the adoption of the state plane coordinate systems. In fact, much has already been done in this direction, notably by

---

NOTE.—This Report was published in June, 1941, *Proceedings*. Discussion on this Report has appeared in *Proceedings*, as follows: October, 1941, by Dorr Viele, Esq.

[9] Associate Engr. (Civ.), U. S. Engr. Office, Rock Island, Ill.

[9a] Received by the Secretary October 16, 1941.

[10] "State Systems of Plane Coordinates: Development of State Grid Systems," by O. S. Adams, *Civil Engineering*, January, 1937, p. 33.

the Federal Board of Surveys and Maps, which, as early as 1936, urged its member organizations to use state plane coordinates wherever practicable.

The state plane coordinate systems were called to the attention of the various district and division engineers of the U. S. Engineer Department by circular letter from the office of the Chief of Engineers, dated May 1, 1936. Incorporated in this letter was the following statement:

> "In so far as may be practicable, it is proposed that consideration be given to the future use of this system of basic controlled coordinates in surveys for river and harbor and flood control projects throughout the Engineer Department where strictly geodetic methods are not warranted, due to the relatively restricted area of the survey."

The U. S. Engineer Office at Philadelphia, Pa., has adopted the New Jersey system of plane coordinates for all of its work within the boundaries of that state.[11] The surveys for the Denison Reservoir Project on the Red and Washita rivers along the Texas and Oklahoma border, by the U. S. Engineer Office at Denison, Tex., were based on the Oklahoma Coordinate System, South Zone.[12] The Pittsburgh (Pa.) Engineer District has conducted numerous large-scale mapping operations in connection with proposed projects for flood protection for the City of Pittsburgh and vicinity. This mapping has been based on state plane coordinate systems.[13]

The question of utilizing state plane coordinates for the surveys in connection with the development of the Upper Mississippi River Nine-Foot Channel Canalization Project arose after geodetic surveys had already been initiated. This project extends from Alton, Ill., to Minneapolis, Minn., a river distance of approximately 658 miles, and embraces parts of five states. The use of state plane coordinates was not considered feasible, since the surveys were geodetic in scope and, furthermore, since the use of five different plane coordinate systems in the surveys and mapping for a single project could only lead to considerable confusion, particularly in the work of land acquisition. Accordingly, geographic coordinates referred to the North American Datum were used for all horizontal control surveys and mapping.

Preliminary surveys and mapping required for studies of the proposed Coralville Reservoir Project on the Iowa River in the Rock Island Engineer District have been based on the Iowa Coordinate System, South Zone, although the system has not as yet (1941) received legislative approval. This project, which covers a reach of about 40 river miles and has a maximum width of about 3 miles, readily lends itself to the use of plane coordinates in all survey and mapping operations. When and if the construction of this reservoir is begun, it is proposed to incorporate the state plane coordinates of monumented corners in land descriptions and plats in general conformance with the recommendations contained in the Joint Committee's Progress Report.

Other agencies of the federal government are making extensive use of the state plane coordinate systems. The Tennessee Valley Authority uses the systems as the basis for various types of property surveys, topographic surveys, and many special surveys.[14] The Bureau of Reclamation, in establishing the specifications for the surveys of the Grand Coulee Project, evolved in 1933–1934 a local rectangular coordinate system for the Columbia River Basin Project. However, late in 1935 it was decided to adopt the North Washington system of state plane coordinates for all future Bureau coordination over the reservoir site.[15]

---

[11] "The Utilization of the Plane Coordinates in New Jersey," by Philip Kissam, U. S. Coast and Geodetic Survey, Geodetic Letter, January, 1937, p. 17.

[12] "Maps, Monuments and Control Surveys," by W. W. Studdert, *ibid.*, p. 27.

[13] "Control Surveys for Flood Protection Projects in the Pittsburgh District," by Thomas J. Mitchell, *Civil Engineering*, May, 1939, p. 277.

There are undoubtedly numerous other instances of the use of state plane coordinate systems by governmental agencies that have not come to the writer's attention. All such use will no doubt assist in the education of the engineering profession, the legal profession, and the laity in the use of state plane coordinates in surveys and land descriptions.

The writer concurs in the Committee's recommendation that it is desirable for a land description to be supplemented by a suitable plat, either as an integral part of the description itself, or by reference to its place of filing in the public records. The former method is recommended, since it makes the plat readily available to all persons having occasion to refer to the instrument in which the description is incorporated. This method is desirable, for instance, in land acquisition by condemnation, where the descriptions and plats may not be filed in the local records until title to the lands passes, often a year or more after the acquisition proceedings are instituted. Copies of the descriptions and plats of all parcels involved are first incorporated in the condemnation petition and become a most important part of it.

As a matter of economy, it is often not feasible to attach a separate plat to each description. A large number of small, contiguous parcels can be incorporated in a single plat with a considerable saving of drafting time. In such cases reference to the plat by number in the descriptions of all tracts shown on it should suffice.

The sample descriptions recommended by the Joint Committee include a minimum amount of survey data in the descriptions themselves. As a general practice the minimizing of the survey data in the written description and making reference to the recorded plat for more particular description appear sound. There are instances, however, in which the use of a plat, either as an integral part of the description or by reference to its place of filing, is not practicable. In such cases a description of the form given as Sample Description 3 in the Committee's Report should be used.

6. Sample Right-of-Way Description Based on a Center Line, by Metes and Bounds, Where No Plat is Available.—An important form of description for right-of-way tracts of uniform width is the description based on the surveyed center line of the right of way. This type of description is frequently used for highway, levee, and other right of ways, and is recommended for such use. Describing the center line rather than the boundaries of the tract makes for brevity in the description as well as for easy retracement of the survey. It is believed desirable to add this form of description to those recommended by the Joint Committee. Sample descriptions, both with and without a plat, are given herewith:

---

[14] "Use of State Plane Coordinate Systems in Property Surveys by the Tennessee Valley Authority," by George D. Whitmore, U. S. Coast and Geodetic Survey, Geodetic Letter, January, 1937, p. 21.

[15] "Combining Geodetic Survey Methods with Cadastral Surveys," by Carl M. Berry, *Transactions*, Am. Soc. C. E., Vol. 105 (1940), p. 1300.

"BLANK RESERVOIR

TRACT NO. 1 BLANK COUNTY, IOWA

A tract of land situated in the County of Blank, State of Iowa, being a strip of land of uniform width of 66.0 ft located in SW$\frac{1}{4}$NW$\frac{1}{4}$ section 2, township 90 north, range 8 west of the 7th principal meridian, bounded on the westerly end by the west line of said section 2 and on the southerly end by the east and west quarter line of said section 2 and being 33.0 ft on each side of the following described center line, all coordinates and bearings being referred to the Iowa Coordinate System, South Zone:

> Beginning on the west line of said section 2 at a point (coordinates: $x = 2{,}499{,}896.4$ ft, $y = 601{,}131.3$ ft) which is north 01° 53′ west, 543.9 ft from the west quarter corner of said section 2; thence north 75° 17′ east, 498.2 ft to a point of tangency; thence 366.4 ft along the arc of a curve to the right, the radius of which is 286.48 ft and the long chord of which bears south 68° 05′ east, 341.9 ft, to a point of tangency; thence 494.5 ft along the arc of a curve to the left, the radius of which is 1,432.40 ft and the long chord of which bears south 41° 20′ east, 492.0 ft, to a point on the east and west quarter line of said section 2 which is south 88° 54′ west, 125.3 ft from the southeast corner of said SW$\frac{1}{4}$NW$\frac{1}{4}$ section 2, containing 2.3 acres, more or less."

Description 6 is complete in itself and is recommended for use in the event reference cannot be made to a plat. The numbers are given only in numerals and are expressed only to the nearest minute in the bearings and to tenths of a foot in the distances. These units reflect the accuracy of the field work and computations and make the use of the term "more or less" unnecessary.

Descriptions based on a center line are often unsatisfactory in that they fail to describe the end bounding lines adequately. In the absence of a definition of such lines it is logical to assume that the tract is bounded at the two ends by lines perpendicular to the center line at its end points. In Description 6 the end bounding lines are stated definitely and this practice is recommended in all center-line type of descriptions.

7. Sample Right-of-Way Description, Including a Plat As Part of Description.—Description 7 is similar to the general part of Description 6 in that it locates the tract by section, township, and range, defines the end bounding lines, and gives the width as well as the area of the strip (see Fig. 2).

The plat incorporated by reference to the file number gives the coordinates of two land corners, as well as of the point of beginning of the tract, and the courses and distances along the center line of the right-of-way tract. The courses are expressed as grid bearings, although they could be stated equally well as grid azimuths. The latter have the advantage of being more readily utilized in the field in making a retracement survey. Bearings, on the other hand, are undoubtedly more readily understood by the abstracter of titles, or title examiner. Both grid north and true north are shown by arrow on the plat, together with the departure of grid north from true north, expressed in

"BLANK RESERVOIR

TRACT NO. 1 BLANK COUNTY, IOWA

A tract of land situated in the County of Blank, State of Iowa, being a strip of land of uniform width of 66.0 ft located in SW¼NW¼ section 2, township 90 north, range 8 west of the 7th principal meridian, bounded on the westerly end by the west line of said section 2 and on the southerly end by the east and west quarter line of said section 2, containing 2.3 acres, more or less, said Tract No. 1 being more particularly shown on a plat designated as file BR-F-1 attached hereto and made a part hereof."

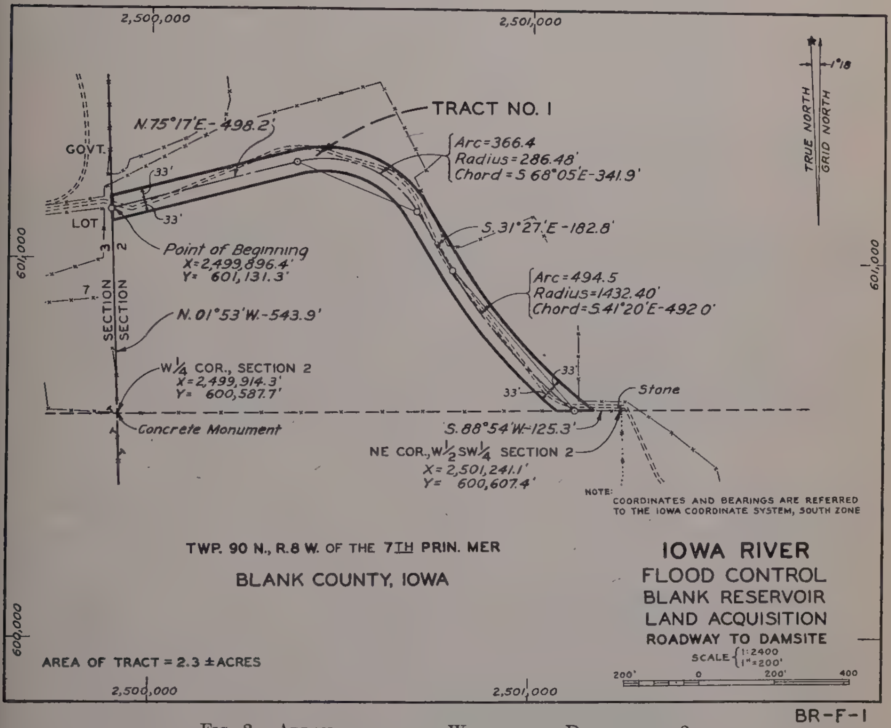


FIG. 2.—ARRANGEMENT AND WORDING OF DESCRIPTION 2

degrees and minutes. These data should be shown on the plat in every case where the relationship can be computed readily.

The use of coordinates for descriptions and plats of boundary prior to official recognition of the coordinate systems through the enactment of enabling laws must necessarily proceed with caution. Two properly executed independent surveys originating at different control stations and touching upon identical points may result in disagreement as to the coordinates of those points unless the two surveys are properly coordinated. Without proper coordination of adjacent or overlapping surveys, identical points might carry different sets of coordinates on recorded plats or descriptions compiled from different surveys, thus creating confusion in the minds of title examiners. In other words, although a point has one and only one set of coordinates, the determination of those coordinates by survey is limited by the accuracy of the survey work. The situation is analogous to that which confronted the U. S. Coast and Geodetic Survey in the development of the horizontal control net over the United States. As the work progressed it became necessary to adjust new arcs of triangulation in order to fit them into the net that was already in existence and already adjusted. Eventually a comprehensive adjustment was required, changing all coordinates except those for one central station. The ideal procedure in the use of state plane coordinates would be to establish an adequate number of adjusted control stations prior to the use of the coordinates in land descriptions that are to be filed for record. However, the benefits to be derived from the use of the systems are great enough to justify beginning such use at the present time in spite of the problems that are certain to arise in the coordination of surveys.

To avoid continued difficulty in boundary locations and in the tracing of land titles, a central bureau in each state for the filing and dissemination of data necessary for the establishment of boundary locations is imperative. This agency might best be in the form of a State Land Court with sufficient authority to exercise control over all surveys and descriptions to be recorded. The results of the Joint Committee's survey to determine what methods have already been put into practice to accomplish this end clearly show the past negligence of most states in the matter. The recommendations contained in Section IV of the Committee's Progress Report point the way toward improvement in the recording and use of survey data, particularly in land descriptions.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

WIND STRESS ANALYSIS BY THE *K*-PERCENTAGE METHOD

Discussion

BY OTTO GOTTSCHALK, ESQ.

OTTO GOTTSCHALK,[19] ESQ.[19a]—For more than a century structural analysis has been almost entirely characterized by abstract mathematical speculation. This is especially true of the eighties. It seems as if this trend was started by specialists to establish—as a mystery—the art of expressing simple relations in endless formulas and equations. Even today this characteristic of structural analysts is admired and emulated the world over. In the United States, writers of textbooks have mitigated the learned excesses, but fundamentally confusing and complicating expressions such as "redundants," "least work," etc., still abound—concepts that are not essential to the type of structure to be designed. The present trend is toward simplified methods of balancing moments or rotations, such as that developed by Professor Cross.[7] This symbolizes a much needed reaction to a study of how a structure acts in nature and of relative efficiencies. The cycles of computation, back and forth, to balance a disturbed equilibrium in a structure are easily perceptible to the analyst. Even these arithmetical devices are unnecessary, however, if the analyst will study, directly, the geometrical deformations that can be observed by the eye.

Professor Witmer states (see heading "Present Methods of Design") that "Of the so-called 'exact' methods, the 'slope deflection method' * * * and the method of end-moment distribution * * * are the most outstanding." This statement is incomplete. A form of visual experiment has been advocated by the writer for many years[20] as the logical basis of all structural analysis. A model composed of flexible splines, subjected to a unit geometrical alteration at any place, bends into curves which are the parts of the influence line for the

---

NOTE.—This paper by F. P. Witmer, M. Am. Soc. C. E., was published in June, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1941, by C. M. Goodrich, M. Am. Soc. C. E.; and October, 1941, by Messrs. Francis L. Castleman, Jr., and Clyde T. Morris.

[19] Buenos Aires, Argentine Republic.

[19a] Received by the Secretary September 9, 1941.

[7] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

[20] *Ibid.*, Vol. 105 (1940), p. 1019.

stress represented by the flexed part. The resistance to rotation at the end of a member, of specific stiffness $K = \frac{I}{L}$, is the relative stiffness[21]

$$S = K\,(1 - 0.5\,m) \ldots\ldots\ldots\ldots (5)$$

This simple expression defines implicitly and automatically the end value of a balancing series and is obtained in a minimum of time.

One of the principal lessons derived from the use of splines is to keep the analysis of structures entirely separate from load and stress computations in order to avoid relapsing, even partly, into the former state of "learned" chaos. To analyze a structure for wind stress assume the upper floor (Fig. 4) to be pushed to the left by a unit distance 1. The columns then assume a curved form whose end tangents intersect the floor levels at distance f and f', as shown. The resistance of any one column to the unit horizontal sway is:[22]

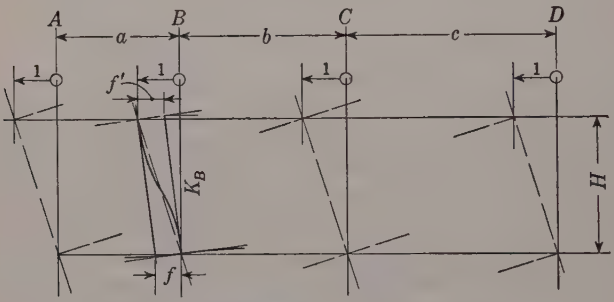


FIG. 4.—DISPLACEMENT "1" OF UPPER FLOOR

$$\Delta P = \frac{6\,E\,K\,(f' - f)}{H^2} \ldots\ldots\ldots\ldots (6)$$

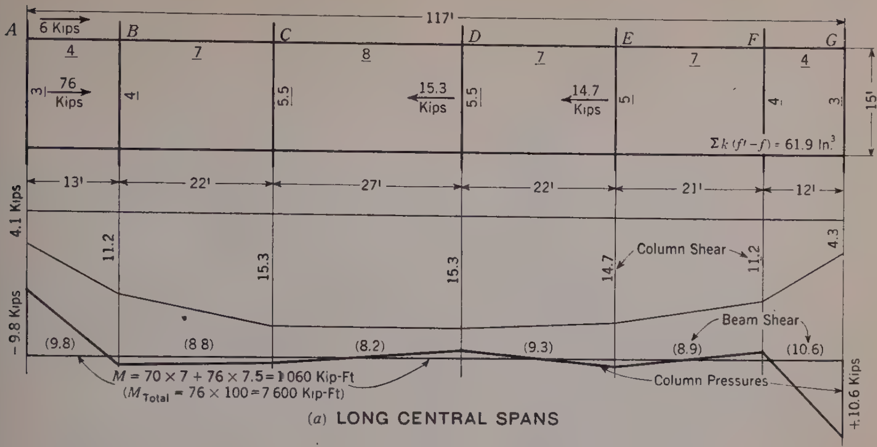
in which, generally, f and f' have opposite signs and P is the force required to produce the unit displacement. Then $\frac{\Delta P}{\Sigma\,\Delta P}$ indicates the share absorbed by that column of the total horizontal thrust P acting at the upper floor. The percentage thus obtained is multiplied by the horizontal thrust P to compute the thrust at each column. The analyst scarcely realized that he has introduced the thrust P into the geometrical analysis of $(f' - f)$ itself; yet that is what is being done all the time.

The ΔP values in every story are relative. Therefore, it is possible to introduce any simplification that affects all members in approximately equal proportions. Generally, in solving Eq. 6 for any single story $\frac{6\,E}{H^2}$ is canceled and (for

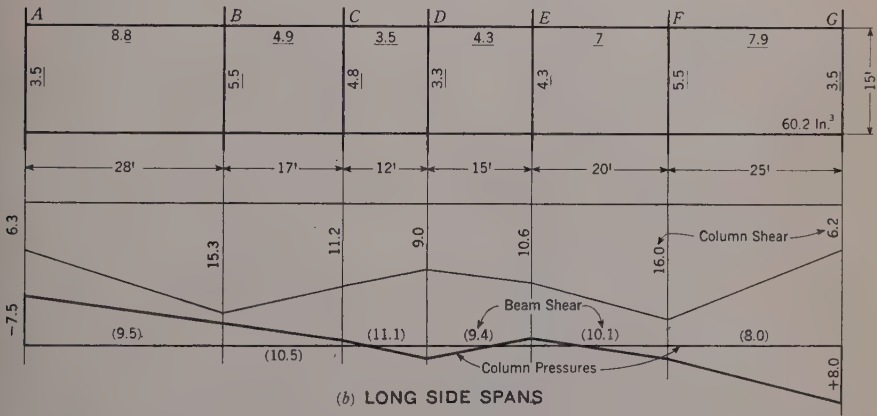
---

[21] "Structural Analysis Based Upon Principles Pertaining to Unloaded Models," by Otto Gottschalk, *Transactions*, Am. Soc. C. E., Vol. 103 (1938), p. 1020.

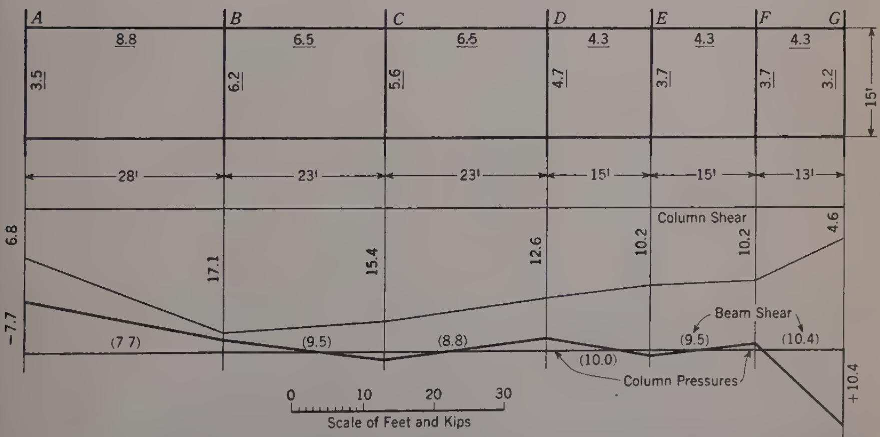
[22] *Ibid.*, Vol. 105 (1940), Eq. 5, p. 1020.



(*a*) LONG CENTRAL SPANS



(*b*) LONG SIDE SPANS



(*c*) SPANS LONG AT ONE END

FIG. 5.—TYPICAL COLUMN SHEAR AND REACTIONS IN A SIX-BAY FRAME, FOR DIFFERENT COLUMN SPACINGS

example) it will be admissible to substitute S by $K = \frac{I}{L}$ for all the structural members, without affecting the results materially. Similarly, in structures generally the columns joining at the upper and at the lower floor of each story will influence the members of that story in equal proportions, and for the purpose of interior analysis, may be omitted.

Fig. 5 shows clearly[23] how the distribution of all these shears is influenced by the relative massiveness of the columns at the sides, at the center, or at one side of the building, proving again that the load distribution is entirely different from that determined with the assumptions made in some of the older methods, as rightly emphasized by Professor Witmer.

Fig. 6 is the three-bay, 10-story bent presented elsewhere by Professor Witmer[24] which the writer has analyzed according to his views. It will be noted that the flexibility of exterior columns is exceptionally great in relation to the remainder of the structure. The analysis can be demonstrated as applied to the story between the sixth and fifth floors (Fig. 7). The girder shears are obtained as follows:

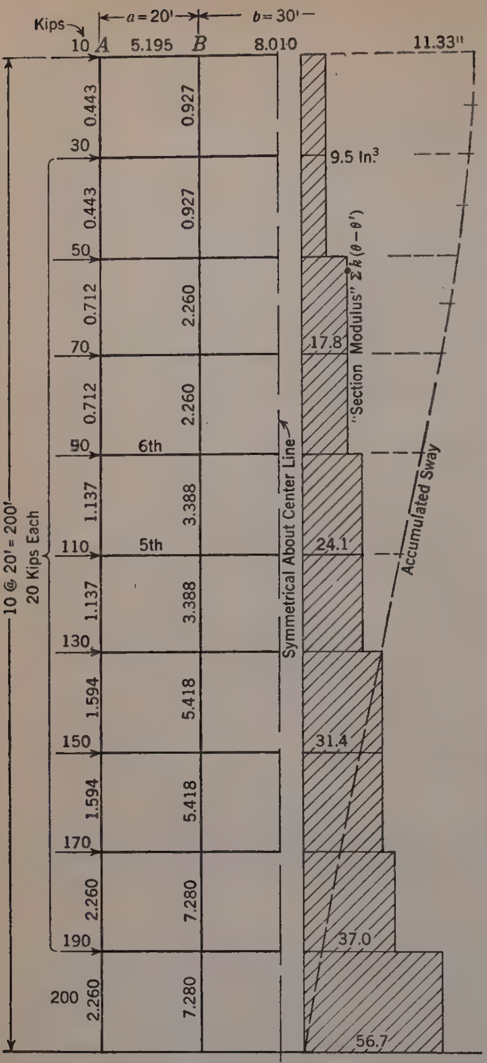


FIG. 6.—THREE-BAY, 10-STORY BENT

$$K_A\,\theta_a = 1.137 \times \frac{5.195}{5.195 + 1.137} = 0.932$$

$$K_B\,\theta'_a = 3.388 \times \frac{5.195}{5.195 + 3.388} = 2.050$$

$$K_B\,\theta_b = 3.388 \times \frac{8.010}{8.010 + 3.388} = 2.382$$

Sum each side of center line 5.364

Total at one floor, $2 \times 5.364 = 10.728$

---

[23] *Transactions*, Am. Soc. C. E., Vol. 105 (1940), Fig. 7, p. 1038; see also Fig. 10, p. 1043, which shows the possibility of obtaining comparative values for the flexibility and sway of a steel building frame.

[24] "Sixth Progress Report of Sub-Committee No. 31, Committee on Steel of the Structural Division, on Wind Bracing in Steel Buildings," *Proceedings*, Am. Soc. C. E., June, 1939, p. 969.

Proportions: $\frac{0.932}{10.728} = 0.087$; $\frac{2.050}{10.728} = 0.191$; and $\frac{2.382}{10.728} = 0.222$.

$$\frac{M_a}{0.5\ M} = 0.087 + 0.191 = 0.278; \qquad S_a = -R_A = \frac{M_a}{10} = 22.24 \text{ kips}$$

$$\frac{M_b}{0.5\ M} = 0.222 + 0.222 = 0.444; \qquad S_b = \frac{M_b}{15} = 23.68$$

Reaction at column B: $R_B = S_b - S_a = +1.44$ kips

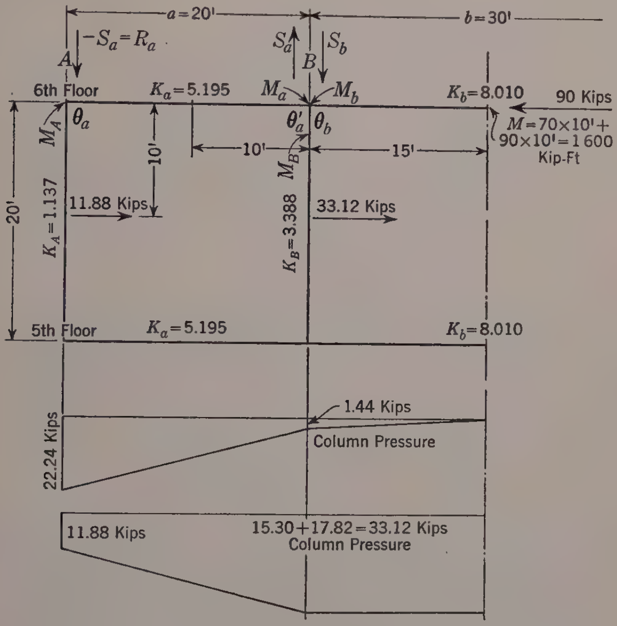


FIG. 7.—COMPUTATION OF STRESSES AT THE SIXTH FLOOR

The shear at the columns between the fifth and sixth floors may be computed as follows:

$$2\ K_A\ \theta_a = \frac{2 \times 1.137 \times 5.195}{5.195 + 2 \times 1.137} = 1.583$$

$$K_B\ \theta'_a = \frac{3.388 \times 5.195}{5.195 + 3.388} = 2.050$$

$$K_B\ \theta_b = \frac{3.388 \times 8.010}{8.010 + 3.388} = 2.382$$

Sum each side of center line 6.015

Total, 2 × 6.015 each at bottom and top = 12.030

Section modulus for this level: $\frac{I}{c} = \Sigma K(\theta - \theta') = 24.060$

Proportions: $\frac{1.583}{12.030} = 0.132$; $\frac{2.050}{12.030} = 0.170$; and $\frac{2.382}{12.030} = 0.198$.

$S_A = 90 \times 0.132 = 11.88\ K$; and $S_B = 90\ (0.170 + 0.198) = 33.12\ K$.

Table 4 shows the bending moments obtained by the foregoing analysis in

TABLE 4.—COMPARATIVE MOMENTS IN A SYMMETRICAL THREE-BAY TEN-STORY BENT (FIG. 6)

(C = Cross Method (Averages); W = Witmer Method; G = Gottschalk Method; D_c = Percentage Difference Between G and C; and D_w = Percentage Difference Between G and W.)

Floor or story	OUTER MEMBERS					INNER MEMBERS				
	C	W	G	D_c	D_w	C	W	G	D_c	D_w
					(*a*) GIRDERS					
10	15.6	13.7	14.8	−5	+8	21.7	22.7	20.6	−5	−9
9	60.5	54.7	59.2	−2	+8	79.5	90.6	82.1	+4	−10
8	110.6	109.4	111.2	+1	+2	182.3	181.3	177.6	−3	−2
7	164.3	164.1	166.8	+1	+2	281.0	271.9	266.4	−5	−2
6	217.0	218.7	222.4	+2	+2	368.7	362.5	355.2	−4	−2
5	273.4	273.4	278.0	+2	+2	453.5	453.1	444.0	−2	−2
4	326.6	328.1	326.4	0	−1	551.5	543.8	547.2	−1	+1
3	379.8	382.8	380.8	0	−1	649.7	634.4	638.4	−2	+1
2	431.8	437.5	430.4	0	−2	729.4	725.0	742.3	+2	+2
1	448.8	440.3	433.1	−3	−2	753.5	729.5	747.0	−1	+2
					(*b*) COLUMNS					
10	15.6	13.7	15.9	+2	+13	33.4	36.4	34.1	+2	−7
9	48.0	41.0	47.7	−1	+14	103.0	109.0	108.3	+5	−7
8	62.3	68.4	62.5	0	−8	187.6	181.7	187.5	0	+3
7	88.6	95.7	87.5	−1	−8	261.3	254.3	262.5	0	+3
6	120.1	123.0	118.8	−1	−3	328.9	326.9	331.2	+1	+2
5	147.0	150.4	145.2	−1	−3	403.0	399.6	404.8	0	+1
4	166.5	177.7	163.8	−2	−7	485.8	472.3	486.8	0	+3
3	196.7	205.1	189.0	−4	−8	558.5	544.9	561.0	0	+3
2	228.4	232.4	222.7	0	−5	621.6	617.6	627.3	+1	+2
1	208.6	207.9	208.4	0	0	597.6	552.2	631.9	....	....
Base	267.9	251.5	255.2	−5	+2	825.9	884.4	804.5	....	....

girders and columns respectively, compared with the average moments of the Cross method and the moments of the Witmer methods. It will be seen that the writer's results are close to the average of the Cross moments and generally between Professor Cross and Professor Witmer.

The lower part of Fig. 7 shows the column pressures and column shears produced by the 90 kips of wind pressure acting at the sixth floor, and by the $M = 1{,}600$ kip-ft of moment, quite distinct again from general, arbitrary assumptions.

Knowing the accumulated column pressures, and their shortening or lengthening in relation to the nearest column, the correction is simple enough to determine by Eq. 6, substituting for H the length L of the connecting beam between columns.[20] Additional important data obtained by the writer[20] are the section modulus and "accumulated sway" in Fig. 6, which establish and define a valuable criterion for the flexibility of different column arrangements in a bent.

Altogether the merits of Professor Witmer's paper and the K-percentage method cannot be questioned, especially in comparison with the results of the older arbitrary methods; but a method, separate from the structure, is not needed if it is analyzed in the manner suggested herein. The three-bay, 10-story bent is most convenient for training purposes and demonstration; yet results obtained with it do not apply to the irregular bents encountered in daily practice, where exterior columns are less flexible compared with the other members of the structure.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

DESIGN AND CONSTRUCTION OF SAN GABRIEL DAM NO. 1

Discussion

BY E. SOUCEK, JUN. AM. SOC. C. E.

E. SOUCEK,[3] JUN. AM. SOC. C. E.[3a]—This discussion is confined to Fig. 24 and one of the author's comments in connection with it. The ranges of heads plotted for each model and the corresponding prototype ranges (in feet) are approximately as follows:

Model	Model heads	Prototype heads
1 : 20	0.50 to 0.85	10 to 17
1 : 100	0.10 to 0.20	10 to 20
1 : 500	0.02 to 0.05	10 to 25

It may be that there was no necessity for discharges corresponding to heads lower than about 10 ft. If this was the case, the tests served their purpose but they do not justify the author's statement (see heading "Spillway") "that the discharge for all models and probably also for the prototype" can be expressed by Eq. 5. This inference should be qualified as to the range of head within which the formula is to be applied. The conformity between the models is excellent in the range of heads compared. An extension of the comparison to lower heads would be equally valuable and interesting.

NOTE.—This paper by Paul Baumann, M. Am. Soc. C. E., was published in September, 1941, *Proceedings*.

[3] Hydr. Engr., The Panama Canal, Balboa Heights, Canal Zone.

[3a] Received by the Secretary October 3, 1941.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

PILE-DRIVING FORMULAS

PROGRESS REPORT OF THE COMMITTEE ON THE BEARING VALUE OF PILE FOUNDATIONS

Discussion

BY MESSRS. HOWARD T. EVANS, WILLIAM G. ATWOOD, DONALD M. BURMISTER, WALLACE E. BELCHER, CLEMENT C. WILLIAMS, AND D. P. KRYNINE

HOWARD T. EVANS,[32] ESQ.[32a]—Working along the lines suggested in 1936 by R. D. Chellis,[20] M. Am. Soc. C. E., the writer has obtained some checks on the use and value of the general formula (Eq. 3) which he believes may be of interest in analyzing the Report under discussion. He has also developed a method of applying the formula on the job which has proved itself in practice.

The first question usually raised in regard to the "complete" formula is whether or not the assumptions on which it is based are sufficiently correct to make it possible to estimate, by this means, the force required to make the ground fail and yield under the tip during driving. Were it possible to insert an accurate spring balance under the tip and read it at each blow of the driving hammer, the general accuracy of the formula could be verified. In effect, this can be accomplished by regarding the pile itself as the spring balance, and measuring its elastic compression.

The next question raised in regard to the formula is whether or not the force thus estimated or measured is the true ultimate bearing value of the pile. Conclusive proof or disproof of the answer to this question can only be obtained by elaborate and expensive tests which it may never be practicable to make. It would be necessary to install and apply long-time load tests to groups of piles, from which all support by side friction had been removed. The ordinary short-time load test is useless because the pile receives frictional support along

---

NOTE.—This Report was published in May, 1941, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1941, by Messrs. G. G. Greulich, C. O. Emerson and D. O. Northrup, Harry J. Engel, and John D. Watson; and October, 1941, by Messrs. Robert D. Chellis, Lazarus White, John G. Mason, Carlton S. Proctor, George Paaswell, and Abraham Woolf.

[32] Care of Supervisor of Shipbuilding, U. S. Navy, Bethlehem Steel Co. (Shipbuilding Div.), Terminal Island, California.

[32a] Received by the Secretary September 15, 1941.

[20] "A Consideration of Pile Driving with Application of Pile Loading Formula," by Robert D. Chellis. *Journal*, Boston Soc. of Civ. Engrs., January, 1941, Vol. XXXVIII, No. 1.

its entire length, a considerable proportion of which often comes from a stratum of silt or soft clay, and has only temporary value. This temporary gripping effect will gradually disappear under a load applied over a period of months or years, and if it has been counted upon for support, serious settlement may result. This side friction from silt or soft wet clay is negligible during driving, where the downward motion is unceasing and steady under a steam hammer; but the pile is "seized" the moment the driving stops. For the very common condition in which it is necessary to penetrate a soft layer and deliver a load to a firm underlying stratum, it does not appear that any better guide to the bearing value of a pile has yet been determined than the tip resistance under the last few blows of the hammer. It is certainly more valuable in this respect than any practicable load test.

An interesting check on the accuracy of the "complete formula" as a measure of this tip resistance was afforded by a job, of which the writer had supervision. At this site, 10-in.-H-42-lb steel piles, 50 ft long, were driven to rock with a single-acting steam hammer having a ram of 7,500 lb and a stroke of 42 in. Assuming 0.9 as the efficiency of the hammer, the energy input was 270,000 in-lb. At twenty blows to the inch, the computed losses were: (1) Impact loss assuming "coefficient of restitution" of 0.32, 66,000 in-lb; (2) loss in compression of head and cap, 40,000 in-lb; (3) loss in elastic compression of pile, 134,000 in-lb; (4) loss in "quake" of earth, 10,000 in-lb; and (5) loss in friction through soft silt, negligible. Subtracting these losses from the input, the energy remaining at the tip was computed as 20,000 in-lb. Dividing this value by 0.05 in. (which was the movement of the pile at each blow) the tip resistance was found to be 400,000 lb.

If this was the true tip resistance, and Eq. 10 is correct, then $\frac{R_d\,L}{E\,A} = 2\,C$ is the "elastic compression as driven" (see Paragraph *A*-4) $= \frac{400{,}000 \times 600}{29{,}000{,}000 \times 12.35} = 0.67$ in. In other words, according to the computations, the pile should shorten and rebound 0.67 in. at each blow.

This effect was measured in the field, near the completion of the driving of a pile, by holding a piece of paper flat against it near its head and tracing (by means of a pencil guided by a rigidly supported straight edge) the actual elastic compression at each blow. The records in Fig. 1 are reproduced from a tracing of the original field sheets. The compression was found, constantly and repeatedly, to be almost exactly $\frac{5}{8}$ in. It requires a little practice to make this field measurement, but it can be done as many times as is necessary to obtain a convincing result.

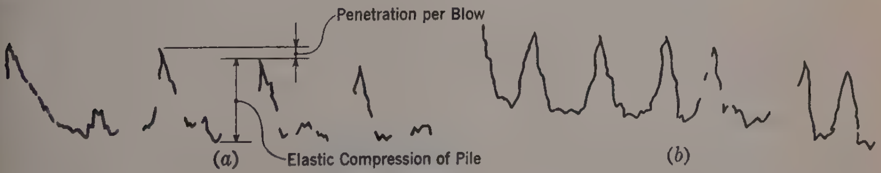


FIG. 1

A further check is necessary, however, before one is justified in accepting the measurement of elastic compression as evidence of the presence of a force of 400,000 lb in the pile. The validity of using the deformation of a pile as a measure of the force by application of the static formula $\left(\text{deformation} = \frac{\text{force} \times \text{length}}{E\,A}\right)$ has been a matter for considerable discussion, since the force is a "sudden" load and not a static one. According to one theory, the temporary deformation under a suddenly applied force should be double the constant deformation under a static load. If this were true, the actual tip resistance in this case would have been shown to be 200,000 lb instead of 400,000 lb. Another method of analysis takes into consideration the travel of a wave of compression along the length of the pile. All these complications are undoubtedly present to some degree.

However, the evidence seems to indicate that, although the simple static formula by no means represents exactly what takes place within the pile, it serves as a means remarkably accurate, even though empirical, of computing the elastic compression and the energy loss. In the case described herein, further verification of the true tip resistance was obtainable by watching these piles closely for yielding of the metal. With a normal yield point of 33,000 lb per sq in., failure should begin at approximately 12.35 × 33,000 = 407,550 lb. Therefore, driving any harder than twenty blows to the inch should produce a noticeable effect on the steel. Such was actually the case, as was observed repeatedly.

If the driving was carried to thirty blows to the inch, yielding near the heads of the piles usually became noticeable. One good pile that had been driven at twenty-five blows to the inch was pulled, and slight "ripples" were found at two points, about midway of the shaft, indicating that yielding had begun. The pile was straight and true nevertheless.

If the "complete" formula is solved for increasingly small penetrations, it will be found that the maximum force, which can be applied to these piles by this hammer under the conditions assumed, is in the vicinity of 410,000 lb. Beyond this, the losses become so great that no more driving force can be developed; that is, at one hundred or three hundred blows to the inch, the tip resistance is little, if any, greater than at thirty blows. This is true for this one set of conditions only. The maximum limit of the force will vary for each length and kind of pile and type of hammer.

The slight yielding of the metal was not damaging to the piles, since driving was stopped before any buckling or serious deformation occurred. The yield point, of course, is far short of the ultimate strength of the metal, and a strain just to the beginning of flow tends simply to raise the yield point for subsequent applications of force in the same direction.

The absence of any tendency to buckle in this case was of special interest as an indication of the very great value of even very soft silt in affording lateral support to the piles, and the needlessness of being too much concerned about the application of ordinary column formulas in their design. One pile struck a hard mass of iron a few feet below the surface of the ground, and before the hammer could be stopped, the steel shaft buckled and was destroyed. In this case there was no side support, and a dramatic example of column failure was afforded. Nothing like this occurred when the pile tips hit the solid rock about 45 ft to 50 ft below the surface, even though the only lateral support was that afforded by a stratum of soft silt. In two or three instances, piles buckled underground when their tips were deflected by boulders or irregularities near the surface of the underlying rock, but in such cases they did not "fetch up" at all, and were definite failures. One of these was pulled and examined, and the battered tip and bent shaft were clear indications of what had happened.

There have been a large number of other instances (both within the writer's direct experience and as reported to him by engineers with whom he has been associated) in which the measurement of the elastic deformation of the pile checked almost exactly the computed deformation. In no case of which the writer is informed has there been any great deviation from it.

The method of applying the "complete formula" discussed herein (which the writer believes to be a new and possibly useful angle of approach) consists of plotting, in advance of the driving, a separate curve for each type and length of pile and each kind of hammer used on a job. This curve shows the tip resistance to be expected for the entire range of values of s. It also shows this resistance graphically in a way that is both interesting and useful.

The x-values in Fig. 2 are shown in terms of blows per inch, and the y-values in terms of the corresponding tip resistances, or ultimate bearing values. Curve A was used as a guide in the driving of the 10-in.-H-42-lb piles, 50 ft long, with a single-acting steam hammer (7,500-lb ram and a stroke of 42 in.), as described in the preceding paragraphs.

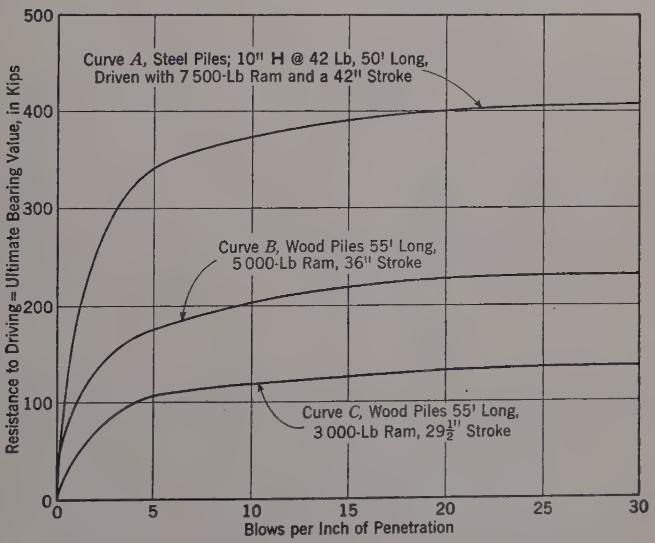


FIG. 2.—PILE-DRIVING CURVES WITH SINGLE-ACTING STEAM HAMMERS

Points along such a curve can be plotted quickly and easily by assuming a series of ultimate bearing values and computing the corresponding energy losses in each case. Deduct the result from the energy input; then divide the residues by the corresponding assumed ultimate bearing values, thereby obtaining the corresponding penetrations in inches. For convenience in field use, convert these into "blows per inch," or "blows per foot." Data presented elsewhere[20] by Mr. Chellis are useful, and in fact necessary, in making these computations.

The method of assuming the ultimate bearing value and solving for the corresponding penetration per blow eliminates the "cut-and-try" procedure required if the reverse process is followed, since the "tip resistance" enters into the computation of all the "C" values in the general formula. The traditional approach comes, of course, from the fact that, in the field, "s" is observed and known, and "R" is the value it is desired to find; but the solution of the formula is much more easy and direct if it is taken the other way around.

The curves plotted as suggested herein will be found to be especially interesting in the way they show what may be expected if driving is carried to refusal. In one case within the writer's experience, in driving 55-ft wood piles with a single-acting hammer (3,000-lb ram and 29½-in. stroke), difficulty was encountered in getting them well seated in the underlying solid stratum, the presence of which was shown by borings. The job superintendent felt sure that, whereas thirty or forty blows to the inch might not be too severe on the piles, driving of one hundred to three hundred blows to the inch certainly was too hard and should not be tolerated under any circumstances. The curve showed otherwise, as it flattened out at about thirty blows to the inch, and indicated clearly that three hundred blows did not stress the piles appreciably more than thirty. Such proved to be the case in actual practice, and several hundred piles were driven hard in this manner until they were at the desired depth, without any sign of injury. The fact was, of course, that the hammer was not large enough. A heavier hammer was not available at first, but when a 5,000-lb single-action hammer (stroke 36 in.) was finally obtained, the piles penetrated to the desired level without difficulty in most cases. The curve for the latter hammer showed, however, that with it, driving to absolute refusal would set up stresses beyond the ultimate compressive strength of the wood, and it was necessary to exercise special care to stop before damage was done to the pile.

The curves plotted for use on this job for the two sizes of hammers are shown as "B" and "C" on Fig. 2. It is desirable that a separate curve or set of curves be plotted for each job. No attempt should be made to develop and use a set of master graphs, as this might tend to discourage the study of the special conditions pertaining to each project.

In conclusion, the writer protests against the development or use of any "pile-driving formula" as such. It is misleading and unsafe to seek a magic combination of terms, in a formula, that will fit any and all cases regardless, and which is supposed to indicate just what load a pile will support. Evidence of this fact is the manner in which "practical" men, for years, have applied the "Engineering News" formula blindly and have stated flatly that such and such a pile was good for exactly so many tons. As long as there is a formula of any kind, this attitude will be encouraged. Instead, the engineer should seek to promulgate the use of a "method of analysis," to be handled by engineers, the same as the design of a girder or truss, for example. To be sure, there are many formulas used in girder and truss design, but there is no one "grand formula" that does it all. The girder designer computes his loading, determines how much fixity he should assume at the supports, and computes the bending moments and shears. Then, taking into consideration the characteristics of the material of which the girder is to be made, he computes the section required. He does not seek to have at hand a "complete formula" into which he can feed all the elements of the design like nickels into a gambling machine, and turn a crank expecting to "hit the jackpot." He proceeds, instead, through a series of steps, each of which he may be assumed to understand thoroughly and which leads logically to the desired result.

The analysis of pile driving should be approached the same way; first, by a careful study of the boring data and soil conditions; then by taking into consideration, one by one, as they occur, all such elements as the type of hammer, material, size and length of pile, quake of the ground, friction of the ground, and the like, using the appropriate formulas and coefficients for each. Thus will the "complete formula" be applied, not as a mystic and marvelous combination of terms, but as a series of steps in a logical method of analysis. Only by such an approach will engineers be able to solve pile-driving problems intelligently.

William G. Atwood,[33] M. Am. Soc. C. E.[33a]—Report *A* presents several formulas of questionable value. It would seem that there are no formulas of general or even local value unless they are treated with good judgment and corroborated by many tests. If that is true, why try to use a formula?

The writer remembers a wharf built under his general direction near St. Nazaire, France, during World War I. The bottom was very soft, and the longest piles that could be obtained were little longer than 100 ft. They were driven to cutoff with single-acting steam hammers (5,000-lb ram and 36-in. stroke) and averaged about 6 in. at the last blow. After the piles had set overnight, it required long pounding to start them again.

Under war emergency it was necessary to take chances, and a safe loading of 300 lb per sq ft was given. This amounted to about 15 tons per pile. After the wharf had been in service under a load of 1,200 lb per sq ft, parts of it settled about 2 ft and then stopped settling. No known formula would have justified the original 300 lb per sq ft, to say nothing of the 1,200 lb per sq ft under which it settled.

The writer also remembers a pile trestle on the Alaska Central Railway where 100-ft piles were driven into glacial silt with a very large penetration under a light drop hammer. This trestle carried trains safely, but no formula would have shown it to be safe.

Another case with opposite results also comes to mind. A foundation of poured-in-place, concrete, mandrel driven piles was constructed under a high buried abutment on the Cleveland (Ohio) Short Line Railway. The piles were from 25 to 35 ft long, driven to refusal and, as the writer remembers, figured for about 15 tons loading. This abutment settled ¾ in. by the time the girders were placed and before there was any traffic over it.

---

[33] Cons. Engr., Winter Park, Fla.

[33a] Received by the Secretary September 20, 1941.

It would seem, with the knowledge now available, that the best the Committee could do would be to make some very general statements as to the unsafeness of using formulas, and the necessity for testing and the exercise of judgment. Engineering may be an exact science, but this is a case in which there is no basis for being exact.

DONALD M. BURMISTER,[34] ASSOC. M. AM. SOC. C. E.[34a]—In view of the limitations of any pile-driving formula and of the uncertainties involved in the successful application to the installation of pile foundations in any given situation, it is believed that Report *B* represents the better practice. Once a formula has been printed, it takes on a more or less authoritative character, and the assumption on which it is based and the limitations in its use tend to be forgotten or overlooked. It is believed that not enough is known, definitely, about the dynamics of pile driving and the way a pile develops its permanent bearing capacity for different subsurface conditions to warrant the inclusion of formulas in the Manual of Engineering Practice at the present time. It seems doubtful that any consistent relationship can exist between them that will be of general application for different types of soil and for the extremely varied subsurface conditions usually encountered in practice at a given site.

According to the concepts of soil mechanics, bearing capacity must be determined in relation to some maximum allowable safe settlement for a given structure. The question may be asked: "Does the settlement of a pile under the design static loading bear any relation to the penetration of the pile under the driving energy of the hammer at the last few blows?" It does not seem likely that piles, driven into soils of varying subsurface character, will stress the soil mass under a static loading in anything like the same manner that the driving of a pile does under the impact of a hammer. It seems quite certain that two piles driven to the same resistance (inches per blow) under the last few blows of a hammer will have quite different settlement characteristics, because of the following differences: (*a*) Total depth of penetration into the soil; (*b*) total energy (total number and distribution of blows) to drive the pile; and (*c*) the question of whether the resistance has been developed (Case 1) gradually and quite consistently along the length of the pile as it is driven; or (Case 2) rapidly only in a resistant stratum into which the point of the pile has been driven.

These differences are indicated by plotting the profile of the variation of the blows per foot to drive the pile and the profile of the total blows. Experience seems to indicate, so far as the settlement of the pile with respect to the soil in which it is embedded is concerned (not due to presence of deeper compressible material below the pile points), that the group effect is likely to be considerably more important for Case 1, where the bearing capacity is developed primarily by frictional resistance along the embedded length of the pile, and may be considerably less important for Case 2, where the bearing capacity is developed primarily by point resistance. In the first case, the settlement of the group may be ten to twenty times, or even more, than that of a single test pile under the same static load per pile; in the second case it may be only of the order of five times as great. This depends on the way the pile load is transmitted to the soil and on the capacity of the soil to take stress—that is, its stress-strain properties. Preliminary model studies of pile foundations for idealized conditions have revealed some interesting facts on this phase of the subject.[35]

---

[34] Asst. Prof., Civ. Eng., Columbia Univ., New York, N. Y.

[34a] Received by the Secretary September 29, 1941.

These aspects of the problem must be considered for each given case. Accurate knowledge of subsurface conditions over the site, complete driving records of a number of piles over the site, and load testing of piles represent the only safe basis for design and installation. With experience and the application of suitable factors based on loading tests and on these other factors, almost any formula can be used as a "yardstick to help the engineer secure reasonably safe and uniform results over the entire job."

WALLACE E. BELCHER,[36] M. AM. SOC. C. E.[36a]—The Report is particularly gratifying in that neither Report *A* nor Report *B* places any reliance on the "Engineering News" formula for determining the safe sustaining power of piles as now being driven. Both sections also place emphasis on the criterion for satisfactory load tests calling for a total net settlement not exceeding 0.01 in. for each ton of applied load. The Report would be of greater value if the advocates of the two sections had seen fit to unite on all such items which they could agree upon, before making separate statements of points on which general agreement could not be reached.

The attempt to introduce a new formula (Report *A*) is of very doubtful value as it is based on the same fundamental data that invalidate the "Engineering News" formula, and it contains an unknown quantity h_0 which is very difficult to arrive at in the case of individual piles and hammers. The tabulation of values of skin friction could well be more complete to include other materials, such as sand, and a more definite description of the material in which the tests were made.

It is the opinion of the writer that additional emphasis should be placed on the redriving of piles after a rest period to be taken in connection with the test loads prescribed. The number of test loads that can be applied is very limited, in any case; whereas, it would be comparatively easy to make a redriving test on a larger number of piles including one pile of each group.

The "Engineering News" formula can give results that are either too high or too low. For instance, in soils such as quicksand and silt, a temporary resistance is offered that may decrease suddenly within a short time as the material adjusts itself, particularly in the presence of ground water. Piles driven on the shore of the Hudson River in New York showed 1 in. of penetration under seven blows of a double-acting steam hammer, but, on the following day, five blows were sufficient. The use of any formula, based on final penetration per blow in such material, will be misleading on the dangerous side, giving apparent safe loads that could disappear overnight.

---

[35] "Model Studies of Pile Foundations," by M. K. Khalifa, *Bulletin No. 7*, Civ. Eng. Research Laboratories, Columbia Univ., New York, N. Y.

[36] Structural Engr., United Engrs. and Constructors, Inc., Philadelphia, Pa.

[36a] Received by the Secretary September 29, 1941.

On the other hand, most soils show an increase of sustaining power by skin friction as the ground sets against and around the pile. The presence of clay facilitates driving, acting as a lubricant while the pile is in motion and tightening up after it begins to rest. When the formula was first developed, driving was done exclusively by the drop hammer method with a sensible lapse of time between blows. The pile had a short rest period between blows and the formula had greater usefulness. Now, with steam hammers being used almost exclusively, the piles and the ground are in practically continuous agitation and the dynamic formula fails to indicate the full value. Engineers have driven many feet of unnecessary piling under such conditions to meet a formula which is not applicable.

Data from records in northern New York are of interest (see Table 12), showing the great variation of safe bearing power, according to the formula, between time of driving and after a rest period. The ground was wet blue clay and fine sand and clay. The piles used were step-taper piles driven with a single-acting hammer having a 5,000-lb ram and a stroke of 36 in., sinking initially 15 ft to 20 ft under the weight of the hammer alone.

TABLE 12.—VARIATION OF SAFE LOAD AS COMPUTED BY "ENGINEERING NEWS" FORMULA

Pile No.	ORIGINAL DRIVING				REDRIVING		
	Date	Depth (ft)	Penetration under last blow	Formula load (tons)	Date	Penetration during redriving	Formula load (tons)
36V6	Dec. 10	35	0.6 in.	26	Dec. 18	6 blows, 0.25 in.	43
36V7	Dec. 10	38	0.55 in.	23	Dec. 18	4 blows, 0.25 in.	43
1B2	Jan. 31	48	0.92 in.[a]	15	Jan. 31[a]	22 blows, 1 ft	23
5H1	Jan. 21	39[b]	12.0 ft[b]	0.1	Jan. 21[b]	17 blows, 2.00 in.	69
38M4	Dec. 12	47.5	0.5 in.[c]	25	Dec. 12[c]	5 blows, 1 in.	50
15B2	Jan. 6	48.58	0.86 in.[d]	16	Jan. 6[d]	7 blows, 1 in.	62

[a] Immediately after changing core, when driving was resumed, it required 15 blows to start the pile moving (formula load, 150 tons). [b] Driven normally to a depth of 39 ft, this pile then sank 12 ft under one blow. Two hours later 15 blows were required to start it moving again (formula load, 150 tons). [c] Five hours later, 7 blows were required to start this pile moving (formula load, 150 tons). [d] After 1 hr, 4 blows were required for a penetration of 1 in. (formula load, 43 tons), then it required 7 blows for the next inch.

A static load test was made for reference, December 18 to December 26, on a pile driven December 11. The pile sank 26 ft under the weight of the hammer and then required 115 blows to drive it to a depth of 40.25 ft with a penetration under the last blow of 0.86 in. (formula safe load, 16 tons). The loading test showed a settlement of 0.009 ft at 30 tons and 0.03 ft at 60 tons which increased to 0.04 ft when pile driving was begun on adjacent footings. Therefore, the pile was entirely acceptable for a working load of 30 tons.

The foregoing examples are given to illustrate the fact that penetration under the blows of a hammer during driving is not a reliable measure of load-sustaining power. An extreme example would be that of a pile jetted into beach sand where no blows whatever are required and yet, after a rest period, the sustaining power of a substantial pile might be limited only by the strength of the pile material.

For a practical procedure the writer suggests that load tests be made on a few piles for control reference and that redriving tests be made in sufficient number to assure the uniformity of conditions or to discover possible variations.[37]

CLEMENT C. WILLIAMS,[38] M. AM. SOC. C. E.[38a]—The Committee is to be commended for their orderly statements of the general significance of pile-driving formulas and for their cautions relative to their use. They might well have stated some of the obvious general characteristics, or axioms, of pile foundations which would aid in interpreting the estimates of bearing capacity derived from such formulas, such as:

(1) The elements involved; namely, the soils, the piles, and the driving are so variable that precision in the estimates cannot be expected. Any pile formula based on principles of dynamics of driving involves assumptions of elasticity and impact that are known to be only approximately true at best. A formula having complicated refinements is not consistent with the nature of the problem.

The plethora of elaborated literature on "soil mechanics" seems to have misled some engineers into the belief that soil is an engineering material with a well defined "mechanics" analogous to hydraulics for water and metallurgy for steel, following mathematical relationships of elasticity and friction. This science is nearer an "ology" that it is an "ics," and only as it makes possible a more precise designation of soils with definite properties does it apply to pile formulas.

(2) The bearing of piles can be estimated only by inference and should not be considered as adequately determined by any one formula. By a comparison of results from dynamics formulas and statics formulas, and by test piles, assigning relative weights to the methods in accordance with the completeness of the information in each process, a reliable value can be obtained. For example, if little is actually known with regard to the soils in contact with the piles, formulas from statics are of little significance, while if the soils information is complete, much weight might properly be attached to statics formulas. Identical soils will yield identical behavior.

(3) The soil stratum at the point of the piles is of primary importance, since, by principles of statics, it must carry the total superimposed load however that load may be communicated to it, either directly by the piles or indirectly through the soil surrounding the piles. Sufficient explorations should be conducted to determine the nature at this stratum.

---

[37] See "Soil Mechanics and Foundations," by Fred L. Plummer and Stanley M. Dore, Pitman Publishing Corp., New York and Chicago, 1940, Chapter 14.

[38] Pres., Lehigh Univ., Bethlehem, Pa.

[38a] Received by the Secretary October 14, 1941.

In considering dynamics formulas, attention should be called to the necessity of adzing the top of the pile to sound wood before making the impact on which calculations are to be based. A large proportion of the energy of the driving hammer may be dissipated in heat if the head of the pile is entirely broomed and this loss of energy tends to vitiate any dynamics formula. For example, in one instance a broomed pile sank about a third as much under a blow as it did after being adzed.

For small penetrations, as in sand, the deformation of the pile itself should be added to the penetration, s, to preserve the theory of the formula. If this is not done, the results for a penetration of $\frac{1}{4}$ in. may be as much as 50% too large. This principle has been confirmed by observations.

The Hiley formula follows dynamics theory more closely than does the "Engineering News" formula, but unless definite information is at hand for the various factors, it may be less trustworthy than a formula with the factors generalized. Although the "Engineering News" formula is little more than empiric, the large amount of observed data accumulated around it renders it a useful rough index of pile capacity. However, its application must be standardized to obtain reliable results and should be checked against other estimates.

The simple statics formula,

$$R_s = A_s f + a\,p \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (44)$$

a being the area of cross section at the point and p the soil bearing capacity at the point, has sufficient utility to warrant its inclusion in the report, chiefly because of the experience that clusters about it. The deformations of the soils under compression at the pile point and in shear by skin friction at the sides are normally of such magnitude that these two sources of resistance are operative simultaneously and are therefore additive. Where the pile rests on rock or other rigid stratum, the pile acts as a column supported along the sides within the lateral resistance of the soil. The reliability of this formula depends on the completeness of the knowledge relative to the soils in contact with the pile. If, through borings, that information is complete, this formula yields dependable values.

The fact that theories of conjugate stresses applicable to an elastic solid do not apply to soils as Rankin assumed should not be held to nullify the principle that the intensity of lateral pressures in soils increases approximately with the depth and hence to justify jettisoning all statics formulas; and the Committee might well have explored the relationships which affect the reliability of this type of formula.

D. P. Krynine,[39] M. Am. Soc. C. E.[39a]—Of the two reports, *A* and *B*, the latter is preferable. It is too concise, however, and probably its practical application will be difficult for an engineer not familiar with the bibliography of the subject. Besides amplification, some small changes, perhaps, are necessary. For instance, at the end of Paragraph *B*-1 the writer would make the following addition: "and, in some occasions, possible non-uniform pressure distribution between individual piles."

---

[39] Research Associate in Soil Mechanics, Dept. of Civ. Eng., Yale Univ., New Haven, Conn.

[39a] Received by the Secretary October 21, 1941.

Mathematics in Report A.—There is too much mathematics in Report *A*. An engineer must have an excellent knowledge of mathematics, which is one of his "tools"; but in all cases, without any exception, mathematical results must be checked against sound engineering judgment based on experience which can be replaced by no theories. This statement is especially true in the case of pile driving.

The Hiley Formula Examined.—The value of k in the Hiley formula (Eq. 4) forms the bulk of the denominator $s + k$, if penetration per blow is small; hence, in this case the value of k controls the value of the dynamic resistance. Therefore, the values of k should be determined with great care.

In Paragraph *A*-8 the value of k is defined as one half of the rebound of the hammer. It equals the sum of $C + C_1 + C_2$, neither of these values depending on the weight of the hammer. It follows that the value of the rebound of the hammer is assumed to be independent of the weight of the hammer.

Furthermore, a question that arises in examining the Hiley formula is the following: Is the value of the dynamic resistance, R_d, as furnished by this formula, a constant for a given pile? In other words, does the dynamic resistance, R_d, depend on the properties of the hammer, particularly on its weight? Personally, the writer is not so sure that the value of R_d does not depend on the weight of the hammer. Therefore, in the following computations both assumptions are made.

In Fig. 3 the accumulated number of blows is plotted against penetration, thus obtaining curves for two H-piles 14 in. by $14\frac{1}{2}$ in. (weight 89 lb per ft).

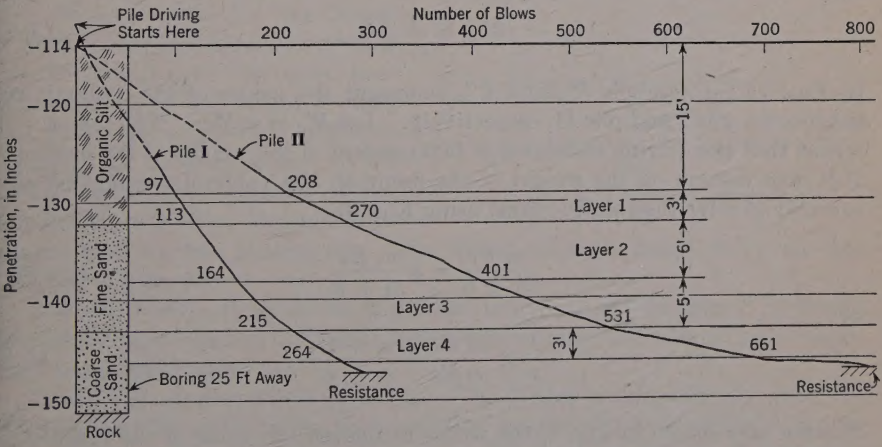


FIG. 3.—NUMBER OF BLOWS (ACCUMULATED) AGAINST TOTAL PENETRATION FOR TWO H-PILES

The total final length of these piles is about 114 ft; but, in driving them through water, a follower about 48 ft long and weighing about 3.5 tons was used. Pile I was driven with a single-acting steam hammer (weight of ram 7,500 lb and a 39-in. stroke). To drive pile II another single-acting steam hammer was used (weight of ram 5,000 lb and a 36-in. stroke). Piles I and II were about 5 ft from center to center so that the soil conditions were practically the same for the two piles. The behavior of piles driven subsequently in the vicinity of piles I and II justified this assumption. Pile II was driven several days after pile I. Both of them passed through organic silt, fine sand, and coarse sand before reaching resistance. Approximate soil conditions, as revealed by a boring 25 ft away, are shown in Fig. 3. A part of the driven depth in Fig. 3 has been subdivided into four layers, one of which corresponds to silt, two to fine sand, and one to coarse sand. Average penetrations s_1 and s_2 of piles I and II, respectively, are to be found in Table 13 (compare also Fig. 3):

TABLE 13.—AVERAGE VALUES OF s_1 AND s_2

Layer	Thickness (ft)	Soil	PILE I[a]		PILE II[a]		Length k (in.)[b]
			Blows	s_1	Blows	s_2	
1	3	Organic silt	16	2.25	62	0.58	1.38
2	6	Fine sand	51	1.41	134	0.54	0.48
3	5	Fine sand	51	1.18	127	0.47	0.36
4	3	Coarse sand	49	0.73	130	0.28	0.25

[a] s_1 and s_2 represent average penetration in inches per blow. [b] Half the rebound 2 k, in inches.

The symbols, E_1 and E_2 (used hereafter for energy furnished by the two hammers), correspond to the values of $e\ W\ h$ as in the Hiley formula. Compute the values of $\frac{W + n^2 P}{W + P}$ for the two cases and call these abstract numbers a_1 and a_2, respectively. Then, applying Eq. 4:

$$R'_d = \frac{a_1 E_1}{s_1 + k} \qquad (45a)$$

and

$$R''_d = \frac{a_2 E_2}{s_2 + k} \qquad (45b)$$

In Eqs. 45 the symbols R'_d and R''_d represent the values of the dynamic resistance for pile I and pile II, respectively. Let $R'_d = x\ R''_d$. The case $x = 1$ means that the driving resistance is independent of the weight of the hammer. If it does depend on the weight of the hammer, the value of x would be presumably greater than unity; thus, using Eqs. 45:

$$\frac{a_1 E_1}{s_1 + k} = x \frac{a_2 E_2}{s_2 + k} \qquad (46a)$$

or

$$\frac{1}{x} \times \frac{a_1 E_1}{a_2 E_2} = \frac{s_1 + k}{s_2 + k} \qquad (46b)$$

For the case shown in Fig. 3, the writer estimated the value of the ratio $\frac{a_1 E_1}{a_2 E_2}$ at 1.85. Placing $x = 1$ in Eq. 46b, values of k for different layers have been computed as shown in Table 13. A simple analysis of Eq. 46b shows that these values are lower limits of the value k. They correspond to the case in which the driving resistance does not depend on the value of the hammer. If it is otherwise, the values of k would be even greater than those in Table 13.

Fig. 3 was prepared on the basis of some pile-driving records of the Connecticut State Highway Department. It is given in this discussion with a twofold purpose: (*a*) To suggest that, in addition to the direct field measurements as advised in Paragraph *B*-10 of the Report, there may be other, perhaps more direct, field methods of estimating the value of *k*; and (*b*) to show how considerable the value of *k* (computed using the Hiley formula) may be in the case of long, heavy H-piles, which practically are not discussed in the Report.

Rebound of the Hammer.—For the case of a rather indeformable hammer and a rather deformable pile the value of 2 *k* practically depends on the elastic deformations of the pile only, as Report *A* rightly states. When a heavy hammer strikes an equally heavy pile, and both of them are elastically deformed, elastic forces may act differently. The writer would like to know definitely whether or not the value of *k* may depend on the properties of the hammer. Another question that arises from simple visual inspection of both Fig. 3 and Table 13 is whether the value of *k* depends on soil characteristics.

Length and Weight of the Pile.—Consider the case shown in Fig. 4. A heavy pile penetrated a stratum of soft material to level A and afterward was driven with a hammer to level B. What is the length *L* to be introduced into Eqs. 10 and 11—the driven length L_1; the length from the earth surface L_2; or perhaps the full length L_3? The same question arises as to the weight *P* of the pile, and to the weight of the follower, if any. All these questions should be clarified in the "Manual," if unfortunately the Hiley formula is recommended for the general use. The writer sincerely hopes, however, that this will not happen.

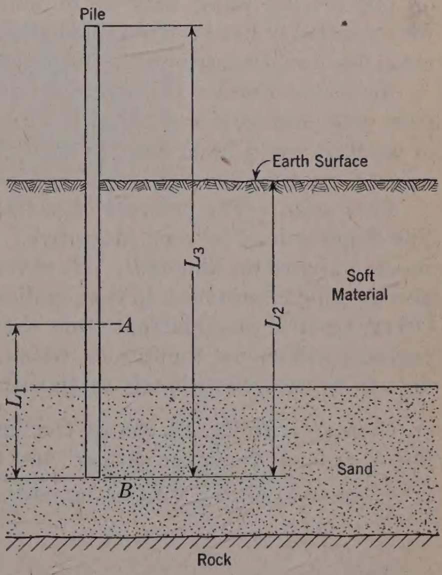


Fig. 4.—Length of a Pile

Battered Piles.—It would be important to give to the profession at least a few rules concerning the driving of both battered piles and vertical piles loaded with horizontal forces.

Local Use of Empirical Formulas.—The writer would not say that the "Engineering News" formula, or some other empirical rule, is entirely inapplicable. It may happen that an engineer working for many years in a locality under given soil conditions has elaborated his own formula or has noticed that in his case the "Engineering News" formula applies. Such an engineer should not be discouraged from using his rules, on the condition, however, that he limit such use to the given locality. As an analogy, the case of forecasting weather may be quoted. Local residents sometimes forecast weather admirably from the signs known to them alone—often better than the "weatherman" does. Nobody, however, would say to these people: "Stop forecasting weather because your method is not scientific." On the contrary, persons depending on weather look for this information, and every engineer who has worked in a primitive country knows it.

Observations Needed.—It is regrettable that the Committee did not furnish more examples from American practice. Examples and references in Report *A* are European, referring mostly to drop hammers and, in some cases, to single-acting steam hammers. Information on the drop hammer is still important, perhaps; but, in addition, it is desirable to give some data on the double-acting steam hammer, which is entirely ignored in the Report.

It would be advisable to take advantage of the enormous statistical material on pile driving which may be obtained throughout the United States, and which probably has been obtained already by the Committee. Such practical examples would contribute to the success of the proposed Manual.

Reinforced Earth.—In some cases as, for instance, in locks or under bridge piers, piles may form a whole with the earth mass. Perhaps simplified methods of analysis could be applied in such cases. This idea was advanced by the writer some time ago.[40]

Conclusion.—The prospect of having a Manual of Engineering Practice on Pile Foundations is very attractive. This Manual, as the writer believes, should be based on Report *B*. However, the text should be tangible and should give genuine information to the practicing engineer, and not mere declarations. Otherwise it is possible that from the two reports, *A* and *B*, the practicing engineer will choose the former, which, though objectionable, gives him some data to be used immediately in his everyday work.

Correction: In Eq. 12, change the "minus" sign to "plus"; and, on page 859, line 5, change "Boulogne sürmer" to "Boulogne-sur-Mer."

---

[40] *Transactions*, Am. Soc. C. E., Vol. 102 (1937), p. 283.